1. Hour exam-Math 361

**Name:**

(1) Let $\Omega = \{1, ..., m\}$ and $P(A) = \frac{|A|}{m}$. Let $A_1, ..., A_r$ be $r$ subset such that

$$\sum_{i=1}^{r} P(A_i) < 1 \; .$$

Show that there exists an element $k \in A_1^c \cap \cdots \cap A_r^c$. (Hint: Try $r = 2$).

(2) Let $X$ be a discrete random variable with

$$Prob(X=0) \ = \ Prob(X=\frac{1}{2}) \ = \ Prob(X=1) \ = \ \frac{1}{3} \ .$$

Calculate the cummulative distribution function $F_X(a)$, the moment generating function $M_X(t)$ and $EX$.

Extra Credit: Let $X_1, ...., X_r$ independent random variables such that $X_1, ..., X_r$ have the same point mass function as $X$. Let $Y_r = X_1 + \cdots + X_r$. Calculate

$$EY_r^2 \ .$$

(3) An urn initially contains one red and one blue ball. At each stage a ball is randomly chosen and then put back adding another one of the same color. For each experiment we assume that every ball is equally likely to be picked. Let $X$ denote the integer valued random denoting the first time a blue color is chosen. For example, if we first draw read and then blue, then $X = 2$.

(a) Let $E_k$ be the event that we draw $k$ red balls successively. Show that

$$P(E_k) \;=\; \frac{1}{k+1}$$

and conclude $\lim_k E_k = 0$.

(b) Show that $P(X = n) \;=\; \frac{1}{n(n+1)}$.

(c) Use (a) to show that $P(X < \infty) = 1$. Use this to show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \;=\; 1\,.$$

(d) Write down the formula for $EX$. We recall that $\sum_n \frac{1}{n+1}$ is $\infty$. What can you say about $EX$.

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**Name:**

(1) Let $\Omega = \{1, ..., m\}$ and $P(A) \;=\; \frac{|A|}{m}$. Let $A_1, ..., A_r$ be $r$ subset such that

$$\sum_{i=1}^{r} P(A_i) < 1\,.$$

Show that there exists an element $k \in A_1^c \cap \cdots \cap A_r^c$. (Hint: Try $r = 2$).

**Solution:** We calculate

$$P((\bigcup_i A_i)^c) = 1 - P(\bigcup_i A_i) \;\geq\; 1 - \sum_i P(A_i) > 0\,.$$

Therefore $P((\bigcup_i A_i)^c)$ and hence there has to be an element in $(\bigcup_i A_i)^c$.

(2) Let $X$ be a discrete random variable with

$$Prob(X = 0) \;=\; Prob(X = \frac{1}{2}) \;=\; Prob(X = 1) \;=\; \frac{1}{3}\,.$$

Calculate the cummulative distribution function $F_X(a)$, the moment generating function $M_X(t)$ and $EX$.

**Solution:**

$$EX = \frac{1}{3}(0 + \frac{1}{2} + 1) = \frac{1}{2}.$$

Also

$$Ee^{tX} = \frac{1}{3}(1 + e^{\frac{t}{2}} + e^t).$$

For $F_X(a)$ we have

$$F_X(a) \begin{cases} 0 & \text{if } a < 0 \\ \frac{1}{3} & \text{if } 0 \le a < \frac{1}{2} \\ \frac{2}{3} \text{ if } \frac{1}{2} \le a < 1 & \\ 1 & \text{else} \end{cases}.$$

**Extra credit:** The second moment of $Y_r = X_1 + ... + X_r$ is given by

$$M''_{Y_r}(0).$$

we have

$$M_{Y_r}(t) = [\frac{1}{3}(1 + e^{\frac{t}{2}} + e^t)]^r.$$

Now, we differentiate

$$M'_{Y_r}(t) = 3^{-r} r[(1 + e^{\frac{t}{2}} + e^t)]^{r-1}(\frac{1}{2}e^{\frac{t}{2}} + e^t)$$

and

$$M''_{Y_r}(t)$$
$$= 3^{-r}\left(r(r-1)[(1 + e^{\frac{t}{2}} + e^t)]^{r-2}(\frac{1}{2}e^{\frac{t}{2}} + e^t)^2 + r[(1 + e^{\frac{t}{2}} + e^t)]^{r-1}(\frac{1}{4}e^{\frac{t}{2}} + e^t)\right).$$

For $t = 0$ this yields

$$EY_r^2 = 3^{-r}\left(r(r-1)3^{r-2}\frac{25}{16} + r3^{r-1}\frac{5}{4}\right) = \frac{25r(r-1)}{144} + \frac{5r}{12}.$$

(3) An urn initially contains one red and one blue ball. At each stage a ball is randomly chosen and then put back adding another one of the same color. Let $X$ denote the integer valued random denoting the first time a blue color is chosen. For example, if we first draw read and then blue, then $X = 2$.

(a) Let $E_k$ be the event that we draw $k$ red balls successively. Show that

$$P(E_k) \;=\; \frac{1}{k+1}$$

and conclude $\lim_k E_k = 0$.

**Solution:** The first probability of red is $1/2$. The next time we have 2 reds and one blue, so the probability is $2/3$, next time $3/4$ and so on. Thus

$$P(E_k) \;=\; \frac{1}{2}\frac{2}{3}\cdots\frac{k}{k+1} \;=\; \frac{1}{k+1}\,.$$

Clearly, $\lim_k 1/(k+1) = 0$.

(b) Show that $P(X = n) \;=\; \frac{1}{n(n+1)}$.

**Solution:** We have $n-1$ times red and the last time blue. For the last time we have to pick 1 blue out of $n-1$ red. Thus

$$P(X = n) \;=\; P(n-1 \text{ times red})P(1 \text{ blue out of } (n+1) \text{ balls }) \;=\; \frac{1}{n}\frac{1}{n+1}\,.$$

(c) Use (a) to show that $P(X < \infty) = 1$. Use this to show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \;=\; 1\,.$$

**Solution:** $X = \infty$ means we only have red balls. Therefore

$$P(X = \infty) \;=\; \lim_k P(E_k) \;=\; 0\,.$$

This implie

$$1 = \sum_{n=1}^{\infty} P(X = n) \;=\; \sum_{n=1}^{\infty} \frac{1}{n(n+1)}\,.$$

(d) Write down the formula for $EX$. We recall that $\sum_n \frac{1}{n+1}$ is $\infty$. What can you say about $EX$.

**Solution:** Formally

$$EX \;=\; \sum_{n=1}^{\infty} nP(X = n) \;=\; \sum_{n=1}^{\infty} n\frac{1}{n(n+1)} \;=\; \sum_{n=1}^{\infty} \frac{1}{n+1} \;=\; \infty\,.$$

2. Hour exam-Math 361

**Name:**

(1) Let $X$ and $U$ be random variables such that $U$ is uniform and for fixed $u$ $X$ is exponential with rate $u$. Find the distribution of $X$. (Hint $\int_0^1 ue^{-ux}du = x^{-2}(1 - e^{-x} - xe^{-x})$. If you can verify this you get extra credit.)

(2) Let $X$, $Y$ positive random variables with joint density function $f_{X,Y}$.

(a) Let $U = \frac{X-Y}{Y}$ and $V = Y$. Show that

$$f_{U,V} = |v| f_{X,Y}(v(u+1), v) .$$

(b) Let $X$, $Y$ be independent, $X$ be $\gamma$-$(\alpha, 1)$ distributed and $Y$ be $\gamma$-$(2, 1)$ distributed. Find the distribution of $\frac{X-Y}{Y}$.

(3) Let $X$ and $U$ be independent. $U$ uniformly distributed over $[0,1]$. We want to show

$$P(X-U>\lambda) \;\leq\; \ln(1+\frac{1}{\lambda})EX \tag{0.1}$$

for all $\lambda > 0$.

(a) Show that for $\lambda > 0$ we have

$$\int\limits_0^1 \frac{1}{\lambda+u}du \;=\; \ln(1+\frac{1}{\lambda}) \,.$$

(b) Let $f_X$ be the marginal density of $X$. Show that

$$P(X-U>\lambda) \;=\; \int\limits_0^1 P(X>\lambda+u)du \,.$$

(Hint write the first term as a double integral.)

(c) Apply the Chebychev inequality and show (0.1).

Practice problems

(1) Let $X$ and $Y$ be positive random variables with joint density function $f_{XY}$. Let $U$, $V$ be given by $U = X-Y$, $V = (X(1+Y))$. Find the joint density for $f_{UV}$.

(2) Let $X$ and $Y$ be independent random variables. We assume that

$$P(Y=1)=p \quad \text{and} \quad P(Y=0)=1-p$$

and that $f_X$ is the density function of $X$.

(a) Show that

$$f_{X-Y}(z) \;=\; pf_X(z+1)+(1-p)f_X(z)\,.$$

(b) Show that

$$P(X-Y>\lambda) \;\leq\; \frac{\lambda+1-p}{\lambda(\lambda+1)}EX\,.$$

Calculate both sides for the $X$ the exponential random variable.

(3) Let $X$, $Y$ be independent exponential random variables. Show that $X-Y$ and $X+Y$ are not independent.

(4) Let $X$ be $N(0,1)$ distributed. Find the density function for $Y = X^3$.

(5) Problem 42 p=294

(6) Problem 44 p=294

(7) Problem 23 p=292

(8) Problem 29 p=293

(9) Problem 3 p=380

(10) Problem 8 p=380

Practice problems

(1) Let $X$ and $Y$ be positive random variables with joint density function $f_{XY}$. Let $U$, $V$ be given by $U = X - Y$, $V = (X(1+Y))$. Find the joint density for $f_{UV}$.

**Solution:** We first calculate the Jacobian

$$\begin{vmatrix} 1 & -1 \\ 1+Y & X \end{vmatrix} = X + 1 + Y \; .$$

Then, we have to calculate for given $u$, $v$ the solution to

$$u = x - y \quad v = x(1+y) \; .$$

This yields $x = u + y$ and

$$v \;=\; (u+y)(1+y) \;=\; u + y + uy + y^2 \;=\; u - \frac{(u+1)^2}{4} + (y + \frac{u+1}{2})^2 \; .$$

This yields

$$y \;=\; -\frac{u+1}{2} + \sqrt{v - u + \frac{(u+1)^2}{4}}.$$

Note that $v \geq u$ is equivalent to

$$x(1+y) - (x-y) \;=\; x(1+y) \geq 0$$

which is certainly true for positive random variables. Thus we picked the right square root. Therefore, we get

$$\begin{aligned} f_{UV}(u,v) \;&=\; \frac{1}{|x+Y+1|} \\ & f_{X,Y}(\frac{u-1}{2} + \sqrt{v-u+\frac{(u+1)^2}{4}}, -\frac{u+1}{2} + \sqrt{v-u+\frac{(u+1)^2}{4}}) \; . \end{aligned}$$

(2) Let $X$ and $Y$ be independent random variables. We assume that

$$P(Y = 1) = p \quad \text{and} \quad P(Y = 0) = 1 - p$$

and that $f_X$ is the density function of $X$.

(a) Show that

$$f_{X-Y}(z) \;=\; pf_X(z+1) + (1-p)f_X(z)\,.$$

**Solution:** By independence we get

$$\begin{aligned} P(X - Y \le a) &= pP(X - Y \le a|Y = 1) + (1-p)P(X - Y \le a|Y = 0) \\ &= pP(X - 1 \le a) + (1-p)P(X \le a) \\ &= pP(X \le a+1) + (1-p)P(X \le a)\,. \end{aligned}$$

Differentiation yields

$$f_{X-Y}(x) \;=\; pf_X(x+1) + (1-p)f_X(x)\,.$$

(b) Show that

$$P(X - Y > \lambda) \;\le\; \frac{\lambda + 1 - p}{\lambda(\lambda+1)}EX\,.$$

**Solution:**

$$\begin{aligned} P(X - Y > \lambda) &= \int_\lambda^\infty f_{X-Y}(x)dx \\ &= \int_\lambda^\infty pf_X(x+1)dx + \int_\lambda^\infty (1-p)f_X(x)dx \\ &= p\int_{\lambda+1}^\infty pf_X(x)dx + (1-p)\int_\lambda^\infty f_X(x)dx \\ &= pP(X > \lambda + 1) + (1-p)P(X > \lambda) \\ &\le p\frac{1}{\lambda+1}EX + (1-p)\frac{1}{\lambda}EX \\ &= \frac{p\lambda + (1-p)(\lambda+1)}{(\lambda+1)\lambda}EX \\ &= \frac{\lambda + 1 - p}{(\lambda+1)\lambda}EX\,. \end{aligned}$$

Calculate both sides for the $X$ the exponential random variable.

**Solution:** We note that

$$EX = \int_0^\infty xe^{-x}dx = -xe^{-x}|_0^\infty - \int_0^\infty -e^{-x} = 1 .$$

Thus we have $\frac{\lambda+1-p}{(\lambda+1)\lambda}$ for the right hand side. For the left hand side, we deduce from $P(X > t) = \int_t^\infty e^{-x}dx = e^{-t}$. Therefore

$$\begin{aligned} P(X - Y > \lambda) &= pP(X > \lambda + 1) + (1 - p)P(X > \lambda) \\ &= pe^{-(\lambda+1)} + (1 - p)e^{-\lambda} = e^{-\lambda}(\frac{p}{e} + 1 - p) . \end{aligned}$$

Thus we get (the not so exciting) inequality

$$e^{-\lambda}(\frac{p}{e} + 1 - p) \le \frac{\lambda + 1 - p}{(\lambda + 1)\lambda} .$$

(3) Let $X$, $Y$ be independent exponential random variables. Show that $X - Y$ and $X + Y$ are not independent.

**Solution:** $U = X - Y$, $V = X + Y$. The jacobian is 2 (or -2). Thus the absolute value is 2. Solving for $x$ and $y$ given

$$u = x - y \quad v = x + y$$

yields $u + v = 2x$, $u - v = -2y$, hence

$$x = \frac{u + 2}{2} \quad y = \frac{v - u}{2} .$$

Hence, as long as $u + v \ge 0$ and $v - u \ge 0$, we find

$$\begin{aligned} f_{X-Y,X+Y}(u, v) &= \frac{1}{2}f_{XY}(\frac{u + v}{2}, \frac{v - u}{2}) \\ &= \frac{1}{2}e^{-\frac{u+v}{2}}e^{-\frac{v-u}{2}} = \frac{1}{2}e^{-v} . \end{aligned}$$

Note that $0 \le u + v$ and $0 \le v - u$ means $|u| \le v$. Now, we have to calculate the marginal distributions

$$\begin{aligned} f_{X-Y}(u) &= \int_{\mathbb{R}} f_{X-Y,X+Y}(u, v)dv = \frac{1}{2}\int_{|u|}^\infty e^{-v}dv \\ &= \frac{1}{2}e^{-|u|} . \end{aligned}$$

Similarly, if $v \geq 0$

$$f_{X+Y}(v) = \int_{\mathbb{R}} f_{X-Y,X+Y}(u,v)du \;=\; \frac{1}{2}\int_{-v}^{v} e^{-v}du$$

$$= ve^{-v} \, .$$

For $v < 0$, we have 0. Taking products we get

$$f_{X-Y}(u)f_{X+Y}(v) \;=\; \frac{1}{2}e^{-|u|}\frac{1}{2}ve^{-v} \;=\; \frac{v}{4}e^{-(v+|u|)} < \frac{1}{2}e^{-v} \, .$$

Thus they are not independent.

(4) Let $X$ be $N(0,1)$ distributed. Find the density function for $Y = X^3$.

**Solution:** We recall the general rule $Y = g(X)$ then for any increasing function $g$ with $g(-\infty) = -\infty$, $g(+\infty) = +\infty$:

$$P(g(X) \leq a) = P(X \leq g^{-1}(a)) \;=\; \int_{-\infty}^{g^{-1}(a)} f_X(x)dx$$

$$= \int_{-\infty}^{a} f_X(g^{-1}(z))\frac{dz}{g'(g^{-1}(z))} \, .$$

Here we use the change of variable $z = g(x)$, $dz = g'(x)dx = g'(g^{-1}(z))dx$. So $dx = \frac{dz}{g'(g^{-1}(z))}$. Differentiation yields

$$f_{g(X)}(a) \;=\; f_X(g^{-1}(a))\frac{1}{g'(g^{-1}(a)))} \, .$$

For $g(x) = x^3$, we find

$$f_{X^3}(x) = \frac{f_X(x^{1/3})}{3(x^{\frac{1}{3}})^2} \;=\; \frac{1}{3\sqrt{2\pi}}x^{-\frac{2}{3}}e^{-\frac{x^{\frac{2}{3}}}{2}} \, .$$

(5) Problem 42 p=294

**Solution:** $f_{XY}(x,y) \;=\; xe^{-x(y+1)}$ for $x, y$ both positive. Using $z = xy$ $dz = xdy$

$$f_X(x) = \int_0^{\infty} xe^{-x(y+1)}dy \;=\; e^{-x}\int_0^{\infty} e^{-xy}xdy$$

$$= e^{-x}\int_0^{\infty} e^{-z}dz \;=\; e^{-x} \, .$$

This gives

$$f_{Y|X}(y|x) \;=\; \frac{f_{XY}(x,y)}{f_X(x)} \;=\; \frac{xe^{-x(y+1)}}{e^{-x}} \;=\; xe^{-xy}\,.$$

This means for fixed $x$, we have

$$P(Y \le a|X = x) \;=\; \int_0^a xe^{-xy}dy \;=\; 1 - e^{-xa}\,.$$

Similarly, we first write $u = x(y+1)$, $du = (y+1)dx$

$$f_Y(y) = \int_0^\infty xe^{-x(y+1)}dx \;=\; \int_0^\infty ue^{-u}\frac{du}{(y+1)^2}\,.$$

Now, integration by parts gives

$$\int_0^\infty ue^{-u}du = u(-e^{-u})|_0^\infty - \int_0^\infty (-e^{-u})du \;=\; 1\,.$$

Therefore

$$f_Y(y) \;=\; (y+1)^{-2}\,.$$

Moreover,

$$f_{X|Y}(x|y) \;=\; \frac{f_{XY}(x,y)}{f_Y(y)} \;=\; (y+1)^2xe^{-x(y+1)}\,.$$

For calculating $f_{XY}$, we note that

$$\begin{aligned}
P(XY \le a) &= \int_{xy\le a} f_{XY}(x,y)dxdy \\
&= \int_0^\infty \int_0^{a/x} xe^{-x(y+1)}dydx \\
&= \int_0^\infty \int_0^{a/x} e^{-xy}xdye^{-x}dx \\
&= \int_0^\infty \int_0^{a} e^{-z}dze^{-x}dx \\
&= \int_0^\infty (1-e^{-a})e^{-x}dx \;=\; (1-e^{-a})
\end{aligned}$$

This means $f_{XY}(a) = e^{-a}$.

(6) Problem 44 p=294

**Solution:** After some serious guessing, I decided that for $acc =$ accidents and $p =$ parameter, we know that the parameters are $\gamma$-$(s, \alpha)$ distributed, i.e.

$$f_p(y) \ = \ \frac{\alpha^s y^{s-1} e^{-\alpha y}}{\Gamma(s)}$$

and most importantly

$$P(acc = n | p = y) \ = \ \frac{y^n e^{-y}}{n!} \ .$$

Therefore, (using $z = (\alpha+1)y$, $dz = (\alpha+1)dy$)

$$\begin{aligned}
P(acc = n) = \int P(acc = n|p = y) f_p(y) dy \ &= \ \int_0^\infty \frac{y^n e^{-y}}{n!} \frac{\alpha^s y^{s-1} e^{-\alpha y}}{\Gamma(s)} dy \\
&= \frac{\alpha^s}{n!\Gamma(s)} \int_0^\infty y^{n+s-1} e^{-(\alpha+1)y} dy \\
&= \frac{\alpha^s}{n!\Gamma(s)(\alpha+1)^{n+s}} \int_0^\infty z^{n+s-1} e^{-z} dz \ = \ \frac{\alpha^s \Gamma(n+s)}{(\alpha+1)^{n+s} n! \Gamma(s)} \ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(acc = n, p \leq T) &= \int_{-\infty}^{T} P(acc = n|p = y) f_p(y) dy \\
&= \int_0^T \frac{y^n e^{-y}}{n!} \frac{\alpha^s y^{s-1} e^{-\alpha y}}{\Gamma(s)} dy \\
&= \frac{\alpha^s}{n!\Gamma(s)} \int_0^T y^{n+s-1} e^{-(\alpha+1)y} dy \ .
\end{aligned}$$

Differentiating, we get

$$\frac{\partial P(acc = n, p \leq T)}{\partial T} \ = \ \frac{\alpha^s}{n!\Gamma(s)} T^{n+s-1} e^{-(\alpha+1)T} \ .$$

Of course we have to divide by $P(acc = n)$ for finding an honest to god probability density. Thus the conditional probability is given by

$$f_{p|acc=n}(T) = \frac{\frac{\alpha^s}{n!\Gamma(s)} T^{n+s-1} e^{-(\alpha+1)T}}{\frac{\alpha^s \Gamma(n+s)}{(\alpha+1)^{n+s} n!\Gamma(s)}}$$
$$= \frac{(\alpha+1)^{n+s}}{\Gamma(n+s)} T^{n+s-1} e^{-(\alpha+1)T} .$$

This all is for the first year. Don't ask me how the recalculate the parameter in the second year.

(7) Problem 23 p=292 $f_{XY}(x,y) = 12xy(1-x)$ for $0 < x, y < 1$. We calculate

$$f_X(x) = \int_0^1 f_{XY}(x,y)dy = \int_0^1 12xy(1-x)dy = 6x(1-x) .$$

Also

$$f_Y(y) = \int_0^1 f_{XY}(x,y)dx = \int_0^1 12xy(1-x)dy$$
$$= 12y \int_0^1 (x - x^2)dx = 12y(\frac{1}{2} - \frac{1}{3}) = 2y .$$

Since $6x(1-x)2y = 12x(1-x)y$, we see that yes they are independent. We find

$$EX = \int_0^1 6x(1-x)dx = 1 ,$$

and

$$EY = \int_0^1 2ydy = 1 .$$

As well as

$$VarY = EY^2 - 1 = \int_0^1 4y^2 dy - 1 = \frac{1}{3} ,$$

and

$$VarX = EX^2 - 1 = 36 \int_0^1 x^2(1-x)^2 dx - 1$$

$$= 36\int_0^1 (x^2 - 2x^3 + x^4)dx - 1 \; = \; 36(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}) - 1$$
$$= \frac{6(10 - 15 + 6)}{5} - 1 \; = \; \frac{1}{5} \, .$$

(8) Problem 29 p=293. Let $X, Y$ random variables and let

$$U \; = \; X^2 Y \quad V \; = \; Y \, .$$

The jacobian is

$$\begin{vmatrix} 2xy & 0 \\ x^2 & 1 \end{vmatrix} \; = \; 2xy \, .$$

The solution to $u = x^2 y$ and $v = y$ is given by $y = v$ and $x = \sqrt{u/v}$ (assuming everything is positive). Thus

$$f_{U,V}(u,v) \; = \; \frac{1}{|2xy|} f_{X,Y}(x,y) \; = \; \frac{1}{|2\sqrt{uv}|} f_{X,Y}(\sqrt{\frac{u}{v}}, v) \, .$$

For independent positive $X, Y$, we find

$$f_{X^2 Y}(u) \; = \; \int_{\mathbb{R}} \frac{1}{|2\sqrt{uv}|} f_X(\sqrt{\frac{u}{v}}) f_Y(v) dv \, .$$

In our particular case

$$f_{I^2 R}(u) = \int_0^1 6\sqrt{\frac{u}{v}}(1 - \sqrt{\frac{u}{v}})\frac{2v dv}{\sqrt{uv}} \; = \; 12\int_0^1 (1 - \sqrt{\frac{u}{v}}) dv$$
$$= 12 - 12\sqrt{u}\int_0^1 v^{-\frac{1}{2}} dv \; = \; 12 - 6\sqrt{u} \, .$$

(9) Problem 3 p=380 We essentially did this in class. Note that we have

$$E|X - Y|^\alpha = \int_{[0,1]^2} |x - y|^\alpha d(x,y) \; = \; \int_0^1 \int_0^y (y - x)^\alpha dx dy + \int_0^1 \int_0^x (x - y)^\alpha dy dx$$
$$= 2\int_0^1 (\frac{(y - x)^{\alpha+1}}{\alpha + 1}|_{x=0}^{x=y}) dy$$
$$= 2\int_0^1 (-\frac{(y - x)^{\alpha+1}}{\alpha + 1}|_{x=0}^{x=y}) dy$$

$$= \frac{2}{\alpha+1}\int_0^1 y^{\alpha+1}dy \;=\; \frac{2}{(\alpha+1)(\alpha+2)}\,.$$

(10) Problem 8 p=380

**Solution:** Don't start thinking about the numbers of places at any table. If $X_i = 1$ if person $i$ has no friends among them and $X_i = 0$ if person $i$ has a friend. Then

$$E\text{number of tables} \;=\; \sum_{i=1}^{N} EX_i\,.$$

However,

$$EX_i \;=\; P(\text{person } i \text{ has no friends among the } i-1) \;=\; (1-p)^{i-1}$$

by independence and the fact that for every pair (person$i$,person$k$) with $k < i$ the probability to be not friend is $(1-p)$. Thus the answer is

$$\sum_{i=1}^{N}(1-p)^i \;=\; \frac{1-(1-p)^N}{p}\,.$$

Practice problems for the final

(1) no 56 p=386

(2) no 58 p=386

(3) Let $(A_n)$ be a sequence of independent events such that

$$P(\bigcap_n \bigcup_{k\geq n} A_k) \;=\; 0\,.$$

(a) Use the well-known inequality $z \geq \ln(1+z)$ for $z = -P(A_k)$ and show

$$-\ln(\bigcup_{n\leq k\leq m} A_k^c) \;=\; \sum_{k=n}^{m} -\ln(1-P(A_k)) \;\geq\; \sum_{k=n}^{m} P(A_k)\,.$$

(b) Recall the Cauchy criterion (for positive real numbers)

$$\sum_k a_k \;<\infty$$

if and only if $\lim_n \lim_{m\geq n} \sum_{k=n}^{m} a_k \;=\; 0$. Use this to show that

$$P(\bigcap_n \bigcup_{k\geq n} A_k) \;=\; 0$$

implies that

$$\sum_n P(A_n)\,\infty\,.$$

(4) Use the CLT (central limit theorem) to show that

$$\frac{16}{17\sqrt{2\pi}4e^8} \;\le\; \lim_n P(X_n > np + 4\sqrt{np(1-p)}) \;\le\; \frac{1}{\sqrt{2\pi}4e^8}\,.$$

where the $X_n$ are $(n,p)$ Bernoulli distributed.

(5) The probability of a coin to appear head is $p$. Let $T_k = j$ is the $k$-th success appears after $j$ experiments. What can you say about

$$\lim_{k\to\infty} P(T_k > \frac{k + 4\sqrt{k(1-p)}}{p})\,.$$

(6) Let $Y$ be a discrete and $X$ be an independent normal distribution. Describe $E(e^{XY}|Y)$ and $E(e^{XY}|X)$.

(7) Let $(X_i(1), ..., X_i(m))_{i\in\mathbb{N}}$ be a sequences of random vectors such that every entry is a $\chi^2$ distributed random variable. Find $n_0$ such that for all $n > n_0$

$$P\left(\left|\frac{1}{n}\sum_{i=1}^n (X_i(1)-1)\right| \le 0.1, ..., \left|\frac{1}{n}\sum_{i=1}^n (X_i(1)-1)\right| \le 0.1\right) > 0$$

Can you find a better estimate for $n_0$ if you assume that all the random variables are independent?

Practice problems for the final

(1) no 56 p=386 **Solution:** We condition on the number $P$ of people in the elevator

$$EX = \sum_{k=0}^{\infty} E[X|P=k] \frac{10^k e^{-10}}{k!} \; .$$

Now, if we have $k$ people we have to calculate

$$E[\max_{i=1,\dots,k} Y_i] \; ,$$

where $Y_i$ are independent uniformly distributed on $\{1, ...., N\}$. Therefore by independence

$$\begin{aligned} P(\max_{i=1,..,k} Y_i > t) &= 1 - P(\max Y_i \le t) \\ &= 1 - P(Y_1 \le t)^k \end{aligned}$$

This yields

$$\begin{aligned} E[\max_{i=1,\dots,k} Y_i] &= \int_0^{\infty} P(\max_i Y_i > t) dt \\ &= \int_0^{N} (1 - P(Y_1 \le t)^k) dt \\ &= N - (\sum_{j=1}^{N-1} P(Y_1 \le j)^k \int_{j \le t < j+1} dt) \\ &= N - \sum_{j=1}^{N-1} (\frac{j}{N})^k \end{aligned}$$

This yields

$$\begin{aligned} EX &= \sum_{k=0}^{\infty} N \frac{10^k e^{-10}}{k!} - \sum_{j=1}^{N-1} \sum_{k=0}^{\infty} (\frac{j}{N})^k \frac{10^k e^{-10}}{k!} \\ &= N - e^{-10} \sum_{j=1}^{N-1} e^{\frac{10j}{N}} \\ &= N - e^{-10} \frac{e^{10} - e^{\frac{10}{N}}}{e^{\frac{10}{N}} - 1} \; = \; \frac{N e^{\frac{10}{N}} - N + 1 + e^{-\frac{10(N-1)}{N}}}{e^{\frac{10}{N}} - 1} \end{aligned}$$

$$= \frac{Ne^{\frac{10}{N}} - (N-1) + e^{-\frac{10(N-1)}{N}}}{e^{\frac{10}{N}} - 1} .$$

Amazing problem-I hope its right. ■

(2) no 58 p=386 **Solution:** (a) We simply add the numbers of flips which end with head and tail. Both of the are geometric distributed and hence

$$EX \;=\; \frac{1}{p} + \frac{1}{1-p} \;=\; \frac{1}{p(1-p)} .$$

(b) If we start with head, then we end with tail and vice versa. Thus

$$P(\text{ends head}) \;=\; 1-p .$$

■

(3) Let $(A_n)$ be a sequence of independent events such that

$$P(\bigcap_n \bigcup_{k\geq n} A_k) \;=\; 0 .$$

(a) Show that

$$-\ln(\bigcup_{n\leq k\leq m} A_n) \;=\; \sum_{k=n}^{m} \ln(1-P(A_k)) \;\geq\; \sum_{k=n}^{m} P(A_k) .$$

**Solution:** By independence

$$P(\bigcup_{k=n}^{m} A_k^c) = \prod_{k=n}^{m} P(A_k^c) .$$

Taking the logarithm we deduce with $\ln(1) = 0$ that

$$-\ln P(\bigcup_{k=n}^{m} A_k) \;=\; \sum_{k=n} m - \ln(1-P(A_k)) \geq \sum_{k=n} mP(A_k)$$

using $-\ln(1-x) \geq x$.

(b) to show that

$$P(\bigcap_n \bigcup_{k\geq n} A_k) \;=\; 0$$

implies that

$$\sum_n P(A_n) \infty .$$

**Solution:** Using the fact that the set $B_n = \bigcap_{k\geq n}$ are decreasing, we deduce from

$$0 \;=\; P(\bigcap_n \bigcup_{k\geq n} A_k) \;=\; \lim_n P(\bigcup_{k\geq n} A_k)$$

that

$$\lim_n \lim_{m\geq n} P(\bigcap_{k\geq n}^{m} A_k^c) \;=\; 1\,.$$

This implies with the previous problem that

$$\lim_n \lim_{m\geq n} \sum_{k=n}^{m} P(A_k)) \;=\; 0$$

This is the famous Cauchy criterion for convergence. ■

(4) Use the CLT (central limit theorem)to show that

$$\frac{16}{17\sqrt{2\pi}4e^8} \;\leq\; \lim_n P(X_n > np + 4\sqrt{np(1-p)}) \;\leq\; \frac{1}{\sqrt{2\pi}4e^8}\,.$$

where the $X_n$ are $(n,p)$ Bernoulli distributed.

**Solution:** $X_n = \sum_{i=1}^{n} Y_i$ where $Y_i$ is an indicator random variable with $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1-p$. Note that $EY_1 = p$, $EY_1^2 = p$ and hence $Var(Y_i) = p(1-p)$. This yields

$$\frac{1}{\sqrt{nVar(Y_1)}}(\sum_{i=1}^{n}(Y_i - p)) \;=\; \frac{1}{\sqrt{np(1-p)}}(X_n - np)\,.$$

The central limit theorem ensures that

$$\lim_n P(X_n - np > \sqrt{np(1-p)}a) \;=\; \int_a^\infty e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}\,.$$

In order to calculate this integral approximately, we note that

$$(\frac{-e^{-t^2/2}}{t})' \;=\; e^{-t^2/2} + \frac{e^{-t^2/2}}{t^2}\,.$$

This implies

$$\int_a^\infty e^{-t^2/2}dt \;\leq\; \int_a^\infty e^{-t^2/2}(1+1/t^2)dt \;=\; e^{-a^2/2}a\,.$$

For $a = 4$, we deduce the assertion. For large $a$ this estimate is even not so bad, because

$$\int_a^\infty e^{-t^2/2}dt = \int_a^\infty \frac{(1+1/t^2)}{1+1/t^2} e^{-t^2/2}dt \geq \frac{1}{1+1/a^2}\int_a^\infty e^{-t^2/2}(1+1/t^2)dt$$
$$= \frac{1}{1+1/a^2} e^{-a^2/2} a \,.$$

Note that $1/(1+1/a^2)$ is close to 1 if $a$ is large. The lower estimate follows. ■

(5) The probability of a coin to appear head is $p$. Let $T_k = j$ is the $k$-th success appears after $j$ experiments. What can you say about

$$\lim_{k\to\infty} P(T_k > \frac{k + 4\sqrt{k(1-p)}}{p}) \,.$$

**Solution:** The stopping time $T_k$ is the some of independent geometrically distributed random variables $T_k = \sum_{j=1}^{k} Y_j$ with

$$P(Y_j = l) = (1-p)^{l-1}p \,.$$

Note that $Y_j$ and $T_1$ have the same distribution and that

$$ET_1 \;=\; 1/p$$

and for $q = (1-p)$

$$\begin{aligned} ET_1^2 &= \sum_{l=1}^{\infty} l^2(1-p)^{l-1}p \\ &= \sum_{l=1}^{\infty} l(l-1)(1-p)^{l-1}p + \sum_{l=1}^{\infty} l(1-p)^{l-1}p \\ &= p(1-p)\frac{d^2}{dq^2}(\sum_{l=0}^{\infty} q^l) + ET_1 \\ &= p(1-p)\frac{2}{(1-q)^3} + \frac{1}{p} \\ &= \frac{2(1-p)+p}{p^2} \;=\; \frac{2-p}{p^2} \,. \end{aligned}$$

Therefore the variance satisfies

$$Var(T_1) \;=\; \frac{2+p}{p^2} - \frac{1}{p^2} \;=\; \frac{1-p}{p^2} \,.$$

Therefore the same estimate as in the previous problem

$$\frac{16}{17\sqrt{2\pi}4e^8} \leq \lim_{k\to\infty} P(T_k > \frac{k+4\sqrt{k(1-p)}}{p}) \leq \frac{1}{\sqrt{2\pi}4e^8}$$

holds. ■

(6) Let $Y$ be Poisson with parameter 1 and $X$ be an independent normal distribution. Describe $E(e^{XY}|Y)$ and $E(e^{XY}|X)$.

**Solution:** We know that $E(e^{XY}|Y) = g(Y)$ where

$$g(y) = E(e^{XY}|Y = y) .$$

By independence

$$E(e^{XY}|Y = y) = E(e^{Xy}) .$$

However, this is the moment generating function and hence

$$E(e^{Xy}) = e^{\frac{y^2}{2}}) .$$

Therefore

$$E(e^{XY}|Y) = e^{Y^2/2} .$$

Of course the same trick works if we interchange the roles of $X$ and $Y$. We only have to know that (see page 363)

$$E(e^{xY}) = e^{(e^x-1)} .$$

Therefore the solution is

$$E(e^{XY}|X) = e^{e^X-1} .$$

■

(7) Let $(X_i(1), ..., X_i(m))_{i\in\mathbb{N}}$ be a sequences of random vectors such that every entry is a $\chi^2$ distributed random variable. Find $n_0$ such that for all $n > n_0$

$$P(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)\right| \leq 0.1, ..., \left|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)\right| \leq 0.1) > 0$$

Can you find a better estimate for $n_0$ if you assume that all the random variables are independent?

**Solution:** Let us first fix $1 \le j \le m$. By the weak law of large numbers we know that

$$P(|\frac{1}{n}\sum_{i=1}^{n}(X_i(j)-1)| > \varepsilon) \;\le\; \frac{\varepsilon^{-2}Var(X_1)}{n}\,.$$

The variance of $X = Y^2$ where $Y$ is gaussian is given by

$$Var(X) \;=\; EY^4 - (EY^2)^2 \;=\; 3-1 \;=\; 2\,.$$

Therefore

$$P(|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)| > \varepsilon \text{ or } \cdots |\frac{1}{n}\sum_{i=1}^{n}(X_i(m)-1)| > \varepsilon) \;\le\; 2m\frac{\varepsilon^{-2}}{n}\,.$$

Thus if $n > 200m$, we see that

$$P(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)\right| \le 0.1, ..., \left|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)\right| \le 0.1) > 0\,.$$

If in addition all the events are independent, we see that

$$P(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)\right| \le 0.1, ..., \left|\frac{1}{n}\sum_{i=1}^{n}(X_i(1)-1)\right| \le 0.1)$$

$$= \prod_{j=1}^{m} P(\left|\frac{1}{n}\sum_{i=1}^{n}(X_i(j)-1)\right| \le 0.1)\,.$$

Then it suffices to know that each term is positive. For this it suffices to take $n > 2\varepsilon^{-2} = 200$. That's better. ∎

Final exam 347-2004

**Name:**

(1) Give an example of subset $C \subset X$, $(X,d)$ a metric space such that
   (a) (5P) $C$ is closed, but not compact.
   (b) (5P) $C$ is totally bounded, but not compact.
   (c) (5P) $C$ is compact but not connected.
   (d) (10P) $C$ is sequentially compact, but not compact.

(2) (30P) Let $(a_n)$ be a sequence such that $\limsup_n |a_n|^{\frac{1}{n}} > 1$, $a_0 = 1$ and

$$\sum_{n=0} a_n \;=\; 3$$

Show that there exists $0 < x < 1$ such that

$$2 \;=\; \sum_n n a_n x^{n-1} \; .$$

Hint: You have to combine two theorems form the course.

(3) Show that the set

$$\{(x,y) \; : x \geq -1, -1 \leq y \leq 1 \; e^x \;\leq\; \cos(x+y)\}$$

is compact. Justify your arguments by introducing some auxilary functions.

(4) (40P) Show Dini's theorem: Let $(f_n)$ be a sequence of continuous functions on $[0,1]$ such that

$$f_n(x) \leq f_{n+1}(x)$$

and the limit function $g(x) = \lim_n g_n(x)$ is continuous. Then $(f_n)$ converges uniformly.

(Hint: For every $x \in C$ find $n(x)$ such that $g(x) < f_{n(x)} + \frac{\varepsilon}{3}$. Then you can find $\delta(x) > 0$ such that $d(x,y) < \delta(x)$ implies $|g(y) - g(x)| < \frac{\varepsilon}{3}$ and $|f_{n(x)}(x) - f_{n(x)}(y)| < \frac{\varepsilon}{3}$ (why ?). Apply compactness and define $n_0 = \max\{n(x_1), ..., n(x_m)\}$. Use $g(y) \geq f_n(y) \geq f_{n(x_i)}$ (which $i$?) to conclude.)

(5) (40P) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ and $a_0$,....,$a_{n-1} \in \mathbb{R}$. Let $f$ be defined by

$$f(x) \;=\; a_i$$

whenever $x_i \leq x < x_{i+1}$. Show that $f$ is integrable and that

$$\int_0^1 f(x)dx = \sum_{i=0}^{n} a_i(x_{i+1} - x_i) .$$

(Hint: Choose $\delta > 0$ small enough and consider the partition $I_\delta$ with points $0 = x_0 < x_1 - \delta < x_1 < x_2 - \delta < x_2 < \cdots x_n - \delta < x_n$. Estimate $\overline{\Sigma}(f, I_\delta) - \underline{\Sigma}(f, I_\delta)$ and don't forget a picture.)

Final exam solution 347-2004

**Name:**

(1) Give an example of subset $C \subset X$, $(X, d)$ a metric space such that
   (a) $C$ is closed, but not compact. Indeed, $C = \mathbb{R}$ is closed but not compact.
   (b) $C$ is totally bounded, but not compact. Indeed, $C = [0, 1)$ is totally bounded, because $C \subset [0, 1]$ and totally bounded subsets of totally bounded sets are totally bounded.
   (c) $C$ is compact but not connected. Indeed, $C = [0, 1] \cup [2, 3]$ is closed and bounded, hence compact and $V = [0, 1] = (-1, 1.5) \cap C$ and $W = [2, 3] = (1.5, 4) \cap C$ are open, satisfy $V \cup W = C$ and are both not empty.
   (d) $C$ is sequentially compact, but not compact. Such a set cannot exists (This is the final version of the characterization theorem).

(2) Show the mean value theorem for differentiable functions. State explicitly the assumptions. (You may use Rolle's Lemma).

**Solution:** MVT: $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists $a < x < b$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} .$$

**Proof:** We defined

$$g(x) = f(x) - f(a) - (x - a)\frac{f(b) - f(a)}{b - a} .$$

Then $g(a) = 0 = g(b)$. $g$ is continuous on $[a,b]$ and differentiable on $(a,b)$ and hence there exists $a < x < b$ such that

$$0 = g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} .$$

(3) (30P) Let $(a_n)$ be a sequence such that $\limsup_n |a_n|^{\frac{1}{n}} > 1$, $a_0 = 1$ and

$$\sum_{n=0} a_n = 3$$

Show that there exists $0 < x < 1$ such that

$$2 = \sum_n n a_n x^{n-1} .$$

Hint: You have to combine two theorems form the course.

**Proof:** Define

$$f(x) = \sum_n a_n x^n$$

By the differentiation theorem for power series ($R > 1$ by assumption)

$$f'(x) = \sum_n n a_n x^{n-1} .$$

The mean value theorem for differentiation provides $x$ such that

$$2 = \frac{f(1)-f(0)}{1} = f'(x) .$$

Indeed, $f(0) = a_0$ and $f(1) = \sum_n a_n = 3$.

(4) Show that the set

$$\{(x,y) : x \geq -1, -1 \leq y \leq 1\ e^x \leq \cos(x+y)\}$$

is compact. Justify your arguments.

**Proof:** We define $g_0(x,y) = e^x$, $g_1(x,y) = -\cos(x+y) = -\cos(h(x,y))$ where $h(x,y)$ is given by $h(x,y) = x+y$. We first note that the functions $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ are obviously continuous. Then $h$ is the sum of continuous functions and hence continuous. $g_1$ is the composition of a continuous function with a continuous function, hence continuous. $g_0$ is the composition of $\pi_1$ with the continuous function and hence continuous. Therefore the set

$$A = \{(x,y) : g_0(x,y) - g_1(x,y) \le 0\} = \{(x,y) : g_0(x,y) - g_1(x,y) > 0\}^c$$

is the complement of an pen set because $g_0 - g_1$ is continuous. The set $A'$ defined in the problem satisfies

$$A' = A \cap [-1,1] \times [-1,0]$$

because for $x > 0$ we have $e^x > 1 \ge \cos(x+y)$. ■

(5) Show Dini's theorem: Let $C$ be a compact subset f a metric space and $(f_n)$ be a sequence of continuous functions on $[0,1]$ such that

$$f_n(x) \le f_{n+1}(x)$$

and the limit function $g(x) = \lim_n g_n(x)$ is continuous. Then $(f_n)$ converges uniformly.

**Solution:** Let $\varepsilon > 0$. For every $x \in C$ there exists $n(x)$ such that

$$g(x) - \frac{\varepsilon}{3} < f_{n(x)}(x)\,.$$

Since $f_{n(x)}$ is continuous there exists $\delta_1(x)$ such that

$$d(x,y) < \delta_1(x)$$

implies

$$|f_{n(x)}(x) - f_{n(x)}(y)| < \frac{\varepsilon}{3}$$

Since $g$ is continuous, there exists $\delta_2(x) > 0$ such that

$$d(x,y) < \delta_2(x)$$

implies

$$|g(y) - g(x)| < \frac{\varepsilon}{3}\,.$$

let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$. Then

$$C \subset \bigcup_{x \in C} B(x, \delta(x))\,.$$

By compactness, there exist $x_1, ...., x_m$ such that

$$C \subset B(x_1, \delta(x_1)) \cup \cdots \cup B(x_m, \delta(x_m)) .$$

Let $n > \max\{n(x_1), ...., n(x_m)\}$. Let $y \in C$. Then there exists $x_i$ such that $d(y, x_i) < \delta(x_i)$. Therefore, we get

$$|g(y) - g(x)| < \frac{\varepsilon}{3}$$

and hence

$$f_n(y) = f_{n(x)}(y) \ \geq \ f_{n(x)} - \frac{\varepsilon}{3} \ \geq \ g(x) - \frac{2\varepsilon}{3} \ \geq \ g(y) - \varepsilon .$$

This implies with $f_n(y) \leq g(y)$ that

$$|g(y) - f_n(y)| < \varepsilon$$

for all $y \in C$ and $n > \max\{n(x_1), ...., n(x_m)\}$. ■

(6) Let $0 = x_0 < x_1 < \cdots < x_n = 1$ and $a_0$,....,$a_{n-1} \in \mathbb{R}$. Let $f$ be defined by

$$f(x) \ = \ a_i$$

whenever $x_i \leq x < x_{i+1}$. Show that $f$ is integrable and that

$$\int_0^1 f(x)dx \ = \ \sum_{i=0}^{n} a_i(x_{i+1} - x_i) .$$

**Solution:** Let $\varepsilon >$ and $\delta < \min_{i=0,...,n} x_{i+1} - x_{i+1}$. Let $I = \{x_1, ...., x_{n-1}$ and $J = \{x_1 - \delta, ..., x_{n-1} - \delta\}$. We consider $I_\delta = I \cup J$. We first observe that

$$\inf_{[x_i, x_{i+1} - \delta]} f \ = \ a_i \ = \ \sup_{[x_i, x_{i+1} - \delta]} f .$$

However,

$$\inf_{[x_{i+1} - \delta], x_{i+1}} f \ = \ \min\{a_i, a_{i+1}\}$$

and

$$\sup_{[x_{i+1} - \delta], x_{i+1}} f \ = \ \max\{a_i, a_{i+1}\} .$$

Therefore, we get

$$\overline{\Sigma}(f, I_\delta) - \underline{\Sigma}(f, I_\delta) = \sum_{i=0}^{n-1} \delta(\max\{a_i, a_{i+1}\} - \min\{a_i, a_{i+1}\})$$

$$\leq \delta n(\max_{i=0,\dots,n} a_i - \min_{i=0,\dots,n} a_i)$$

Hence $\delta < \frac{\varepsilon}{n(\max_{i=0,\dots,n} a_i - \min_{i=0,\dots,n} a_i)}$ provides

$$\overline{\Sigma}(f, I_\delta) - \underline{\Sigma}(f, I_\delta) < \varepsilon \; .$$

In particular, we get

$$\underline{\Sigma}(f, I_\delta) \leq \int_0^1 f \leq \overline{\Sigma}(f, I_\delta) \; .$$

The same estimate as above yields

$$|\sum_{i=1}^{n} a_i(x_{i+1} - x_i) - \underline{\Sigma}(f, I_\delta)| \;\leq\; \delta n(\max_{i=0,\dots,n} a_i - \min_{i=0,\dots,n} a_i)$$

The assertion follows. ∎