Cavalieri's symbolism

In order to calculate $\int_0^a x^2 dx$ Cavalieri used the following symbolism for $k = 2$. Let $z = (a/2 - x)$.

$$\begin{aligned} a^3 = \sum_0^a a^2 &= \sum_0^a (x+y)^2 = \sum_0^a x^2 + \sum_0^a y^2 + 2\sum_0^a xy \\ &= 2\sum_0^a x^2 + 4\sum_0^{a/2}(a/2 - z)(a/2+z) \\ &= 2\sum_0^a x^2 + 4\sum_0^{a/2} a^2/4 - 4\sum_0^{a/2} z^2 \\ &= 2\sum_0^a x^2 + a^3/2 - 4\sum_0^{a/2} x^2 \\ &= 2\sum_0^a x^2 + a^3/2 - \frac{4}{8}\sum_0^a x^2 , \end{aligned}$$

This gives

$$a^3 = 4\sum_0^a x^2 - \frac{8}{8}\sum_0^a x^2 = 3\sum_0^a x^2 .$$

This should be read as

$$\int_0^a x^2 dx = \frac{a^3}{3} .$$

The rigorous version of these manipulations has to use the Riemann sums

$$\int_0^a x^2 dx = \lim_n \frac{a}{n}\sum_{k=1}^n \left(\frac{ak}{n}\right)^2 = a^3 \lim_{n\to\infty}\sum_{k=1}^n \left(\frac{k}{n}\right)^2 .$$

The corresponding calculation goes as follows with

$$z_k = \frac{1}{2} - k/n = \frac{1}{2}(\frac{\frac{n}{2}-k}{n/2})$$

$$\begin{aligned} 1 = \frac{1}{n}\sum_{k=1}^n 1 &= \frac{1}{n}\sum_{k=1}^n (\frac{k}{n} + \frac{n-k}{n})^2 \\ &= \frac{1}{n}\sum_{k=1}^n (\frac{k}{n})^2 + \frac{1}{n}\sum_{k=1}^n (\frac{n-k}{n})^2 + 2\frac{1}{n}\sum_{k=1}^n \frac{k}{n}\frac{n-k}{n} \\ &\sim \frac{2}{n}\sum_{k=1}^n (\frac{k}{n})^2 + 4\frac{1}{n}\sum_{k=1}^{n/2}(1/2 - z_k)(1/2 + z_k) \end{aligned}$$

$$\sim \frac{2}{n}\sum_{k=1}^{n}(\frac{k}{n})^2 + \frac{1}{2} - \frac{1}{n}\sum_{k=1}^{n/2}\left(\frac{\frac{n}{2}-k}{n/2}\right)^2 .$$

With the change of variable $n/2 - k$ to k we get

$$1 \sim \frac{4}{n}\sum_{k=1}^{n}(\frac{k}{n})^2 - \frac{1}{n/2}\sum_{k=1}^{n/2}(\frac{k}{n})^2 .$$

However, both terms

$$\frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^2$$

and

$$\frac{1}{n/2}\sum_{k=1}^{n/2}(\frac{k}{n})^2$$

are close to

$$\int_0^1 x^2 dx$$

and hence

$$\int_0^1 x^2 dx = \frac{1}{3} .$$

For the integral $\int_0^a x^3 dx$ Cavalieri used a similar (but simpler) argument. First

$$a^4 = \sum_0^a a^3 = \sum_0^a (x+y)^3 = 2\sum_0^a x^3 + 6\sum_0^a x^2 y .$$

In order to the second one, we have

$$\begin{aligned} a^4 = a\sum_0^a a^2 &= a(\sum_0^a x^2 + \sum_0^a y^2 + 2\sum_0^a xy) \\ &= 2a(\sum_0^a x^2) + 2\sum_0^a xy(x+y) \\ &= \frac{2}{3}a^4 + 4\sum_0^a x^2 y \end{aligned}$$

Hence we have

$$\frac{a^4}{12} = \sum_0^a x^2 y .$$

This gives

$$a^4 = 2\sum_0^a x^3 + \frac{a^4}{2}$$

and thus

$$\frac{a^4}{4} = \sum_0^a x^3$$

corresponding to

$$\int_0^1 x^3 dx = \frac{1}{4}.$$

Problem: Give a more rigorous argument for the last identity using the same algebraic tricks.

Remark: The first step was done as follows

$$\begin{aligned} 1 = \frac{1}{n}\sum_{k=1}^n 1 &= \frac{1}{n}\sum_{k=1}^n (\frac{k}{n} + \frac{n-k}{n})^3 \\ &= \frac{1}{n}\sum_{k=1}^n (\frac{k}{n})^3 + \frac{1}{n}\sum_{k=1}^n (\frac{n-k}{n})^3 \\ &\quad + 3\frac{1}{n}\sum_{k=1}^n (\frac{k}{n})^2 \frac{n-k}{n} + 3\frac{1}{n}\sum_{k=1}^n (\frac{n-k}{n})^2 \frac{k}{n} \end{aligned} \tag{0.1}$$

$$\sim \frac{2}{n}\sum_{k=1}^n (\frac{k}{n})^3 + 6\frac{1}{n}\sum_{k=1}^n (\frac{k}{n})^2 \frac{n-k}{n}. \tag{0.2}$$

Here is the solution:

$$1 = \frac{1}{n}\sum_{k=1}^n 1 = \frac{1}{n}\sum_{k=1}^n (\frac{k}{n} + \frac{n-k}{n})^2 = \frac{1}{n}\sum_{k=1}^n (\frac{k}{n})^2 + \frac{1}{n}\sum_{k=1}^n (\frac{n-k}{n})^2 + + \frac{2}{n}\sum_{k=1}^n \frac{k}{n}\frac{n-k}{n}(\frac{k}{n} + \frac{n-k}{n})$$

$$\sim \frac{2}{n}\sum_{k=1}^n (\frac{k}{n})^2 + \frac{4}{n}\sum_{k=1}^n (\frac{k}{n})^2 \frac{n-k}{n}.$$

Using the Riemann sums for the first term we get

$$\frac{1}{3} \simeq \frac{4}{n}\sum_{k=1}^n (\frac{k}{n})^2 \frac{n-k}{n},$$

i.e.

$$\frac{1}{12} \simeq \frac{1}{n}\sum_{k=1}^n (\frac{k}{n})^2 \frac{n-k}{n}.$$

Plugging this into (0.2) we get

$$\frac{1}{2} \simeq \frac{2}{n}\sum_{k=1}^n (\frac{k}{n})^3$$

and hence, in the limit

$$\int_0^1 x^3 dx = \frac{1}{4}.$$