First Midterm Exam -September 29, 2008

1) (10 P) State the comparison lemma and indicate briefly how it is used in the method of exhaustion.

Solution: Let m_1, m_2 be magnitudes of the same size and s_1, s_2 be magnitudes of the same size and $p_n \leq m_1$, $q_n \leq m_2$ be such that

$$(p_n : q_n) \;=\; (s_1 : s_2)$$

for all $n \in \mathbb{N}$ and

$$m_1 - p_n < 2^{-n} m_1 \quad \text{and} \quad m_2 - p_n < 2^{-n} m_2 \;.$$

Then $(m_1 : m_2) = (s_1 : s_2)$.

This is applied in showing that area of circles c_1, c_2 of radii r_1, r_2 satisfies

$$(a(c_1) : a(c_2)) \;=\; (r_1^2 : r_2^2)$$

by showing that for corresponding similar polygons P_n, Q_n inscribed in c_1, c_2 we have $(a(P_n) : a(Q_n)) = (r_1^2 : r_2^2)$ and the approximation assumption is satisfied.

(1) In the following problems make a sketch of the main construction or argument solving the problem. A small picture with few explanation is enough. The best three explanation will constitute a maximum of 30P.

Example: For the area calculation of the area inscribed in a spiral, one divides the angle into equal parts and compares each angular sector with the area segments of a circle containing the spiral segment and with the area segment of a circle which is contained in the spiral segment.

(a) The are of a circle can be exhausted by the area of polygons inscribed in it. Why is the difference small?

Let P_n a regular polygon and P_{n+1} regular polygon obtained by adding midpoints. Let Δ_n a constituting triangle which leads to two new triangles Δ_{n+1}^1 and Δ_{n+1}^2 and satisfies

$$a(C_\theta) - a(\Delta_{n+1}^1 \cup \Delta_{n+1}^2) \;\leq\; \frac{1}{2} a(C_\theta) - a(\Delta_n)$$

where (C_θ is the circle segment corresponding to Δ_n.

(b) The volume of a pyramid only depends on the height and the base area.

Starting with a pyramid with base triangle Δ and height H one constructs two prism and to pyramids by halving all the sides. Then the sum of the volume of the two pyramids is smaller than the sum V of the volumes of the prisms and moreover

$$V =$$

(c) The area of an ellipse with width b and height a in πab.

Let us assume $a < b$. Here one uses a circle with radius b and compares the trapezoids obtained from a regular polygon inscribed in the large circle which are then pushed down to trapezoids in the ellipse by keeping the same x-axis, but the y coordinate lies the ellipse. The ration of area of these two kind of trapezoids is a/b.

(d) The surface area of a sphere is $4\pi r^2$.

Let D be a disc with radius r. Let P_n be a polygon inscribed in D. Then one considers the body of revolution obtained from P_n. The surface area is a sum of mantle surface areas of cones and frustrae. It turns out that the heights of these cones and frustrae are proportional to their heights using the surface areas for frustrae.

(2) (60P) Show that the mantle (or lateral) surface area of cylinder with radius r and height h is $2\pi rh$. (Guideline: 1) Calculate the surface instead of the mantle area. 2) Use polygons P_n inscribed the disc and polygons Q_n containing the disc. Then consider the three dimensional prisms obtained from both polygons and don't forget the base and the top. In order to show that the surface areas for the collection of prims HP_n and HQ_n are close, it is useful to know that $r'/r \leq (r'/r)^2$ for all ratios $r/'r > 1$).

Solution: Let $P_n \subset D \subset Q_n \subset D'$ so that D is the circle of radius r and D' is a circle of radius r'. We know that if n is large we may make r'/r smaller than every number $\alpha > 1$.

Let us now calculate the full surface area of HP_n, the 3D-object obtained from connecting two parallel P_n with distance H. Then clearly

$$sa(HP_n) \;=\; 2a(P_n) + hl(P_n) \;,$$

because the mantle pieces are just rectangles with height h and the length is given by the length of the parts of the triangles lying on the circle. By Archimedes postulate we get

$$sa(PH_n) \;\leq\; sa(HD) \;\leq\; sa(HQ_n)$$

and

$$sa(HQ_n) \;=\; 2a(Q_n) + hl(Q_n) \;=\; 2a(P_n)(r'/r)^2 + hl(P_n)(r'/r) \;\leq\; (r'/r)^2 sa(HP_n) \;.$$

Since r'/r can be chosen arbitrary close to 1 we get

$$sa(HD) - sa(PH_n) < \varepsilon$$

for a suitable choice of n. Also

$$2\pi r - l(P_n) < \varepsilon \quad \text{and} \quad 2a(D) - 2a(P_n) < \varepsilon$$

and hence

$$sa(HD) \;=\; 2\pi r^2 + 2\pi rh \;.$$

Subtracting top and bottom we get

$$2\pi hr \;.$$

Problem: Calculate the length (circumference) of half of a spiral, given by

$$S \;=\; \{\alpha\theta e^{i\theta} : 0 \leq \theta \leq \pi\} \;.$$

(Guideline: Use approximating triangles (inside and outside) by dividing the angle π into equal parts. You are allowed to use what you know about $\tan(\frac{\pi}{n})$ if n is big.)

Solution: We partition the angle between 0 and π into n pieces $\theta_k = \frac{\pi k}{n}$ and consider the points on the spiral

$$p_k \;=\; \alpha\theta_k e^{i\theta_k} \;.$$

Let $P(n)$ be the polygon obtained from connecting these points. Keeping in ming that p_n is on the x-axis, we deduce from Archimedes postulate that

$$\sum_{k=1}^{n} |p_k p_{k+1}| \;\leq\; l(S) \;,$$

where here $l(S)$ is the length of S without the x-axis. Following the idea from the book we also define

$$q_k \;=\; \alpha\theta_{k+1} e^{i\theta_k} \;.$$

Then S is contained in the polygon connecting these points and moreover, the intersection with the x-axis differs only by $\frac{\pi\alpha}{n}$ and hence

$$l(S) \;\leq\; \frac{\pi\alpha}{n} + \sum_{k=1}^{n} |q_k q_{k+1}| \;.$$

Note however that $|q_k q_{k+1}| = |p_k p_{k+1}|$ and hence

$$\sum_{k=1}^{n} |p_k p_{k+1}| \;\leq\; l(S) \;\leq\; +\frac{\pi}{n} + \sum_{k=2}^{n+1} |p_k p_{k+1}| \;.$$

This is what I was expecting from you!

Therefore it suffices to calculate these sums. In order to this right we have to consider the distance d_k of the point p_k from the line Op_{k+1}, O being the origin. Let

$$r_k \;=\; \frac{\pi k}{n}$$

this distance from p_k to the origin. Then we have

$$\frac{d_k}{r_k} \;=\; \sin(\pi/n) \;.$$

Hence we get

$$\begin{aligned} |p_k p_{k+1}|^2 &= d_k^2 + (r_{k+1} - r_k \cos(\pi/n))^2 \\ &= r_k^2 \sin(\pi/n)^2 + r_{k+1}^2 + r_k^2 \cos^2(\pi/n) - 2r_{k+1} r_k \cos(\pi/n) \end{aligned}$$

$$= r_k^2 + r_{k+1}^2 - 2r_{k+1}r_k\cos(\pi/n)$$
$$= (r_{k+1} - r_k)^2 + 2r_kr_{k+1}(1 - \cos(\pi/n))$$
$$= (\frac{\pi\alpha}{n})^2 + 2\frac{k(k+1)\pi^2\alpha^2}{n^2}(1 - \cos(\pi/n))$$
$$= (\frac{\pi\alpha}{n})^2(1 + 2k(k+1)(1 - \cos(\pi/n)))$$

This yields

$$\sum_{k=1}^{n} |p_kp_{k+1}| \;=\; \pi\alpha\frac{1}{n}\sum_{k=1}^{n}\sqrt{1 + 2k(k+1)(1 - \cos(\pi/n))}\;.$$

And no, Archimedes had no chance for calculating this one, only to note that the length is proportional to α. The only evident thing is the lower estimate

$$\sum_{k=1}^{n} |p_kp_{k+1}| \;\geq\; \pi\alpha\;.$$

Another way to obtain a lower estimate is to replace

$$|p_{k+1}p_k|$$

by

$$|p'_kp_k| \quad p'_k \;=\; r_ke^{i\theta_{k+1}}\;.$$

By comparison with the circle we know that

$$|p'_kp_k| \;\geq\; \beta r_k$$

for every $\beta < 1$ provided n is large enough. Again we find

$$\sum_{k=1}^{n} |p'_kp_k| \;\geq\; \beta\sum_{k=1}^{n}\frac{\alpha\pi k}{n} \;\geq\; \beta'\frac{\pi\alpha}{2}\;.$$

Thus this method gives a worse estimate

$$l(S) \;\geq\; \frac{\pi\alpha}{2}\;.$$