Cavalieri's calculus for $k = 4$

I will use the correct notation and derive three inequalities.

1)

$$\begin{aligned} a^5 = \int_0^a a^4 dx \ &= \ \int_0^a (x + (a-x))^4 dx \\ &= 2\int_0^a x^4 dx + 8\int_0^a x^3(a-x)dx + 6\int_0^a x^2(a-x)^2 dx \end{aligned}$$

2)

$$\begin{aligned} a^5 = a\int_0^a a^3 dx \ &= \ a\int_0^a (x + (a-x))^3 dx \\ &= 2a\int_0^a x^3 dx + 6a\int_0^a x^2(a-x)dx \\ &= 2a\frac{a^4}{4} + 6\int_0^a x^2(a-x)x dx + 6\int_0^a x^2(a-x)^2 dx \ . \end{aligned}$$

This gives

$$\frac{a^5}{2} \ = \ 6\int_0^a x^3(a-x)dx + 6\int_0^a x^2(a-x)^2 dx \ .$$

3) Then the 1/2-trick with $z = a/2 - x$: We also use the substitution $x = y/2$,

$$\begin{aligned} \int_0^a x^3(a-x)dx &= \int_0^{a/2} x^2(a/2-z)(a/2+z)dx + \int_{a/2}^a x^3(a-x)dx \\ &= \int_0^{a/2} x^2(a^2/4 - z^2)dx + \int_0^{a/2} x(a-x)^3 dx \\ &= a^2/4\int_0^{a/2} x^2 dx - \int_0^{a/2} x^2(a/2-x)^2 dx \\ &\quad + a^3\int_0^{a/2} x dx - 3a^2\int_0^{a/2} x^2 dx + 3a\int_0^{a/2} x^3 dx - \int_0^{a/2} x^4 dx \\ &= \frac{a^2}{4}\frac{(a/2)^3}{3} - \frac{1}{2 2^2 2^2}\int_0^a y^2(a-y)^2 dy \\ &\quad + a^3\frac{(a/2)^2}{2} - a^2(a/2)^3 + \frac{3a}{4}(a/2)^4 - \frac{1}{32}\int_0^a y^4 dy \\ &= a^5\left(\frac{1}{96} + \frac{1}{8} - \frac{1}{8} + \frac{3}{64}\right) - \frac{1}{32}\int_0^a x^2(a-x)^2 dx - \frac{1}{32}\int_0^a x^4 dx \end{aligned}$$

Let us multiply this with 6×32. Then we get

$$192\int_0^a x^3(a-x)dx + 6\int_0^a x^2(a-x)dx \ = \ 11a^5 - 6\int_0^a x^4 dx \ .$$

We have three equations with three unknowns. Combining the first two 1)-2) we get

$$a^5/2 \;=\; 2\int_0^a x^4 dx + 2\int_0^a x^3(a-x)dx\ . \tag{0.1}$$

Combining the second and the third we get

$$\begin{aligned} 11a^5 - 6\int_0^a x^4 dx &= 192\int_0^a x^3(a-x)dx + a^5/2 - 6\int_0^a x^3(a-x)dx \\ &= a^5/2 + 186\int_0^a x^3(a-x)dx\ . \end{aligned}$$

Hence

$$21a^5 \;=\; 12\int_0^a x^4 dx + 2\times 186\int_0^a x^3(a-x)dx\ ,$$

and

$$7a^5 \;=\; 4\int_0^a x^4 dx + 124\int_0^a x^3(a-x)dx\ . \tag{0.2}$$

Then (0.2)-2(0.1) yields

$$6a^5 \;=\; 120\int_0^a x^3(a-x)dx$$

and thus

$$a^5 \;=\; 20\int_0^a x^3(a-x)dx\ .$$

Finally back into (0.1) we get

$$a^5/2 \;=\; 2\int_0^a x^5 dx + \frac{2}{20}a^5$$

and therefore

$$\int_0^a x^5 dx \;=\; a^5/4 - a^5/20 \;=\; \frac{a^5}{5}\ .$$