1 Completeness

Claim 1: Let X, Y be Banach spaces. Then $I(X,Y)$ is complete.
Proof: First we prove that

$$\|T\| \;\le\; i(T)\;.$$

Indeed, let $x \in X$ and $y^* \in Y^*$, then

$$|y^*(Tx)| \;=\; |tr(T(y^* \otimes x))| \;\le\; i(T)\,\|y^* \otimes x\| \;\le\; i(T)\,\|y^*\|\,\|x\|\;.$$

Taking the supremum over all x and y^*, this assertion follows from Hahn-Banach. To show the completeness, we consider (T_n) such that

$$i(T_{n+1} - T_n) \;\le\; 2^{-n}\;.$$

Let

$$T \;=\; T_1 + \sum_n T_{n+1} - T_n\;.$$

For $x \in X$, we have

$$\sum_n \|(T_{n+1} - T_n)(x)\| \;\le\; \sum_n i(T_{n+1} - T_n)\,\|x\| \;<\; \infty\;.$$

Hence T is well-defined and for every $S = \sum_{j=1}^m y_j^* \otimes x_j$, we deduce from the continuity of the y_j^*'s

$$\begin{aligned} |tr(TS)| \;&=\; |\sum_{j=1}^m y_j^*(T(x_j))| \;=\; \lim_n |\sum_{j=1}^m y_j^*(T_n(x_j))| \\ &\le\; \lim_n i(T_n)\,\|S\| \;\le\; (i(T_1)+1)\,\|S\|\;. \end{aligned}$$

Hence, T is integral. The same argument shows that

$$\begin{aligned} |tr((T-T_n)S)| \;&=\; \lim_m |tr((T_m - T_n)S)| \\ &\le\; \lim_m i(T_m - T_n)\;\|S\| \;\le\; 2^{1-n}\,\|S\|\;. \end{aligned}$$

Hence, T is the limit with respect to the i-norm. □

Claim 2: V_p is complete.
Proof: Let (x^n) be a sequence such that

$$\left\|x^{n+1} - x^n\right\|_{v_p} \;\le\; 2^{-n}\;.$$

We note that for all $k \in \mathbb{N}$

$$\begin{aligned} |x_k| &= |x_k - x_1| + |x_1| \leq 2^{1-\frac{1}{p}}(|x_k - x_1|^p + |x_1|^p)^{\frac{1}{p}} \\ &\leq 2^{1-\frac{1}{p}} v_p(x) \,. \end{aligned}$$

Hence,

$$x_k = x_k^1 + \sum_n x_k^{n+1} - x_k^n$$

is a well-defined scalar. Let $n \in \mathbb{N}_0$ and $y_k = x_k - x_k^n$ where $x_k^0 = 0$. Let (k_j) be a subsequence. Then, we have

$$\begin{aligned} &\left(|y_{k_1}|^p + \sum_{j=1}^{\infty} |y_{k_{j+1}} - y_{k_j}|^p \right)^{\frac{1}{p}} \\ &= \sup_N \left(|y_{k_1}|^p + \sum_{j=1}^{N} |y_{k_{j+1}} - y_{k_j}|^p \right)^{\frac{1}{p}} \\ &= \sup_N \lim_m \left(|x_{k_1}^m - x_{k_1}^n|^p + \sum_{j=1}^{N} |(x_{k_{j+1}}^m - x_{k_{j+1}}^n) - (x_{k_j}^m - x_{k_j}^n)|^p \right)^{\frac{1}{p}} \\ &= \sup_N \lim_m v_p(x^m - x^n) \leq 2^{1-n} \,. \end{aligned}$$

For $n = 0$ the last estimate has to be replaced by $1 + v_p(x^1)$. Taking the supremum over all subseqeunces, we deduce $x \in V_p$ and $x - x^n$ converges to 0 with respect to the v_p norm. □