Exam 1-Real Analysis

Name:

Problems 1,2,3,4 are obligatory. Problem 5 is extra credit (and you will need extra credit for an A^+).

(1) (15P) Given an example of a closed and bounded set in a metric space which is not compact.

(2) (15P) State the Baire category theorem.

(3) (15P) Show that the $D = \{(x,y) \ : \max(|x|,|y|) = 1\}$ has measure 0 in $\mathbb{R}^2$ with respect to the Lebesgue measure in $\mathbb{R}^2$.

(4) (45P) On $X = \{0,1\}^{\mathbb{N}}$ we recall

$$I_{\varepsilon_1,....,\varepsilon_n} \;=\; \{(\delta_1,\delta_2,...,\delta_n,\delta_{n+1},....) \;:\; \delta_1 = \varepsilon_1,....,\delta_n = \varepsilon_n\}$$

and the measure

$$\mu(I_{\varepsilon_1,....,\varepsilon_n}) = 2^{-n} \ .$$

Let $\phi : X \to [0,1]$ defined by

$$\phi(\delta_1,\delta_2,...) \;=\; \sum_{i=1}^{\infty} \delta_i 2^{-i} \ .$$

Show that for every borel measurable set $E \subset [0,1]$

$$m(E) \;=\; \mu(\phi^{-1}(E)) \ .$$

(Hint: Define a certain collection of subset of $[0,1]$, show that it is a σ-algebra and that it contains the duadic intervals, for extra credit you can also show that this works indeed for Lebesgue measurable sets).

(5) (a) (10P) Let C be a subset of a metric space X and $x_0 \in X$. Show that C is closed whenever $C \cap B(x_0, R)$ is closed for every $R > 0$.

(b) (15P) Let $p(t) = a_0 + ta_1$ and $q(t) = b_0 + tb_1$. Show that

$$\max\{|a_0 - b_0|, \frac{1}{2}|a_1 - b_1|\} \le d(p, q) \le |a_0 - b_0| + |a_1 - b_1| \, .$$

Here the distance is taken in $C[0, 1]$ with respect to the uniform distance.

(c) (25P) Let us consider the polynomials $p_k(t) = t^k$. Show that

$$\{a_0 p_0 + a_1 p_1 \ : \ a_0, a_1 \in \mathbb{R}\}$$

is a closed subset of $C[0, 1]$ (with respect to the uniform norm). (Hint: use compactness and a)and b).

1. Given an example of a closed and bounded set in a metric space which is not compact.

   Solution: The unit sphere in the d_{SNCF} metric, The unit ball $B(0,1)$ in $C[0,1]$.
2. State the Baire category theorem. **Solution:** Let (O_n) be open dense sets in complete metric space. Then $\bigcap_n O_n$ is dense.
3. Show that the $D = \{(x,y) \ : \max(|x|,|y|) = 1\}$ has measure 0 in $\mathbb{R}^2$ with respect to the Lebesgue measure.

   Solution: By additivity it suffices to show that the sets $\{\pm 1\} \times [-1,1]$ and $[-1,1] \times \{\pm 1\}$ have measure 0. For this we observe that

   $$\{\pm 1\} \times [-1,1] \subset [\pm 1 - \varepsilon, \pm 1 + \varepsilon) \times [-1, 1+\varepsilon)$$

   And hence

   $$m(\{\pm 1\} \times [-1,1]) \ \le \ 2\varepsilon(2+\varepsilon) \to_{\varepsilon \to 0} 0 \, .$$

   The other parts are similar.
4. (a) Let C be a subset of a metric space X and $x_0 \in X$. Show that C is closed whenever $C \cap B(x_0, R)$ is closed for every $R > 0$.

   Solution: Let $(x_n) \subset C$ such that $x = \lim_n x_n$. This implies $\lim_n d(x_n, x_0) = d(x, x_0)$ is finite. we may find $R = 2d(x,x_0)$ and n_0 such that $d(x_n, x_0) < R$ for all $n \ge n_0$. By assumption $x = \lim_{n \ge n_0} x_n \in B(x_0, R) \cap C$. Thus in C.

   (b) Let $p(t) = a_0 + ta_1$ and $q(t) = b_0 + tb_1$. Show that

   $$\max\{|a_0 - b_0|, \frac{1}{2}|a_1 - b_1|\} \ \le \ d(p,q) \ \le \ |a_0 - b_0| + |a_1 - b_1| \, .$$

   Solution: The upper estimate is easy. $p(0) - q(0) = a_0 - b_0$ implies that

   $$|a_0 - b_0| \ \le \ |p(0) - q(0)| \ \le \ d(p,q) \, .$$

   Moreover,

   $$|a_1 - b_1| \le |a_1 - b_1 + a_0 - b_0| + |a_0 - b_0| \ \le \ |p(1) - q(1)| + |p(0) - q(0)| \ \le \ 2d(p,q) \, .$$

   There you go.

   (c) Let us consider the polynomial $p_k(t) = t^k$. Show that

   $$\{a_0 p_0 + a_1 p_1 \ : \ a_0, a_1 \in \mathbb{R}\}$$

is a closed subset of $C[0,1]$ (with respect to the uniform norm). (Hint: use compactness and a)

Solution: By $a)$ and $b)$ it suffices to show that

$$C_R \;=\; \{a_0p_0 + a_1p_1 \;:\; |a_0| + |a_1| \;\le\; R\}$$

is closed. However, $f : \mathbb{R}^2 \to C[0,1]$ defined by $f(a_0,a_1) = a_0p_0 + a_1p_1$ is continuous and $\{(a_0,a_1) : |a_0| + |a_1| \le R\}$ is closed and bounded. Heine-Borel implies the assertion.

(5) On $X = \{0,1\}^{\mathbb{N}}$ we recall

$$I_{\varepsilon_1,....,\varepsilon_n} \;=\; \{(\delta_1,\delta_2,...,\delta_n,\delta_{n+1},....) \;:\; \delta_1 = \varepsilon_1,....,\delta_n = \varepsilon_n\}$$

and the measure

$$\mu(I_{\varepsilon_1,....,\varepsilon_n}) = 2^{-n} \;.$$

Let $\phi : X \to [0,1]$ defined by

$$\phi(\delta_1,\delta_2,...) \;=\; \sum_{i=1}^{\infty} \delta_i 2^{-i} \;.$$

Show that for every measurable set $E \subset [0,1]$

$$m(E) \;=\; \mu(\phi^{-1}(E)) \;.$$

(Hint: You may use the information from the model problems. Define a set, show that it is a σ-algebra and that it contains the duadic intervals).

Solution: We first consider a duadic point $x = \sum_{i=1}^{n} \varepsilon_i 2^{-i}$ and $J_{x,n}$ defined by

$$J_{x,n} \;=\; [x, x + 2^{-(n+1)}] \;.$$

Then $\phi^{-1}(J_{x,n}) = I_{\varepsilon_1,...,\varepsilon_n}$. Thus for every $J_{x,n}$ we have

$$m(J_{x,n}) \;=\; \mu(\phi^{-1}(J_{x,n})) \;.$$

Now, let Σ be the collection of subset of $[0,1]$ such that

$$m(E) \;=\; \mu(\phi^{-1}(E)) \;.$$

Note that

$$E, F \in \Sigma \Rightarrow F \setminus E \in \Sigma \tag{0.1}$$

$$m(F\backslash E) \;=\; m(F) - m(E) \;=\; \mu(\phi^{-1}(F)) - \mu(\phi^{-1}(E)) \;=\; \mu(\phi^{-1}(E)^c) \;=\; \mu(\phi^{-1}(F\backslash E)) \;.$$

Now, we observe that for a countable disjoint union (E_j) with $E_j \in \Sigma$ we have

$$m(\bigcup_j E_j) \;=\; \sum_j m(E_j) \;=\; \sum_j \mu(\phi^{-1}(E_j)) \;=\; \mu(\phi^{-1}(\bigcup_j E_j))\,.$$

Thus Σ is a σ-algebra if we can show that Σ is closed under finite unions. Let $E_1, E_2 \in \Sigma$. Then $E_1 \cup E_2 = E_1 \cup E_2 \setminus E_1$. Thus (0.1) and the above implies the assertion. Finally, we note that for every interval $(a,b) \subset [0,1]$ we have

$$(a,b) \;=\; \bigcup_{J_{x,n}\subset(a,b)} J_{x,n}$$

because duadic points are dense. Then

$$[a,b) \;=\; \bigcap_n (a-\frac{1}{n}, b)\,.$$

Thus the Borel algebra of $[0,1]$ is contained in Σ. If $A \subset [0,1]$ has measure 0, then

$$A \subset E$$

and E is in the borel algebra and has measure 0. Thus

$$\mu(\phi^{-1}(A)) \;\le\; \mu(\phi^{-1}(E)) \;=\; 0\,.$$

Hence $\phi^{-1}(A)$ has measure 0 and A is in Σ. Since every measurable set is the union of a borel set and a set of measure 0, we are done.

Exam 2

Name: For the following problems we assume that (Ω, Σ, μ) is a σ-finite measure space.

(1) (30P) We consider $X = \ell_1^n$. Show that $X^* = \ell_\infty^n$ using the isomorphism which associates to a sequence $x = (x_1, ..., x_n)$ the linear functional

$$\phi_{(x_1,....,x_n)}(a_1, ..., a_n) \;=\; \sum_{i=1}^n x_i a_i \; .$$

Solution: Let $\phi : \ell_1^n \to \mathbb{R}$ be a linear functional. We define

$$a_k \;=\; \phi(e_k) \; .$$

Let $\varepsilon_k = 1$ if $a_k \geq 0$ and $a_k = -1$ if $a_k < 0$. Then we have

$$|a_k| \;=\; |\phi(\varepsilon_k e_k)| \;\leq\; \|\phi\| \|\varepsilon_k e_k\| \;\leq\; \|\phi\| \; .$$

This yields

$$\sup_k |a_k| \;\leq\; \|\phi\| \quad \text{and} \quad \phi(\lambda_1, ..., \lambda_n) \;=\; \sum_k a_k \lambda_k \; .$$

Conversely, we consider $a = (a_1, ..., a_n) \in \ell_\infty^n$ and

$$\phi_a(\lambda_1, ..., \lambda_n) \;=\; \sum_k a_k \lambda_k$$

Then

$$\|\phi_a(\lambda_1, ..., \lambda_n)\| \;=\; |\sum_k a_k \lambda_k| \;\leq\; \sup_k |a_k| \sum_k |\lambda_k| \; .$$

This shows that ϕ_a is a continuous linear functional with

$$\|\phi_a\| \;\leq\; \sup_k |a_k| \; .$$

■

(2) (30P) Let $f : \Omega \to \mathbb{R}$ be an integrable function. Show that

$$\lim_{\mu(E) \to 0} \int_E |f| d\mu \;=\; 0 \; .$$

(Hint: Use approximation by simple functions and the σ-continuity of the measure.)

Solution: Let us consider a simple function $g = \sum_i r_i 1_{F_i}$ first. Then

$$\int_E |g| \;=\; \sum_i |r_i| \mu(F_i \cap E) \;\leq\; \sum_i |r_i| \mu(E)$$

converges top 0 for $\lim \mu(E) = 0$. Now, we consider an arbitrary integrable function f. Let $\varepsilon > 0$ and g a simple function such that

$$\int |f-g| < \frac{\varepsilon}{2} \, .$$

Then there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies

$$\int_E |g| < \frac{\varepsilon}{2} \, .$$

This yields

$$\int_E |f| = \int_E |f-g| + \int_E |g| \;\le\; \int |f-g| + \int_E |g| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon \, .$$

■

(3) (30P) Let (f_n) and f be integrable functions. Assume that f_n converges to f. Show that

$$\lim_n \int |f_n| d\mu \;=\; \int |f| d\mu \quad \text{if and only if} \quad \lim_n \int |f-f_n| d\mu \;=\; 0 \, .$$

(Hint: $|f_n - f| \le |f_n| + |f|$ and the practice problem comes in handy for one implication.)

Solution: $\Longrightarrow$: We define $g_n = |f_n| + |f|$ and Then g_n converges to $g = 2|f|$ everywhere. By assumption

$$\lim_n \int g_n \;=\; \int g$$

By practice problem we deduce from $|f_n - f| \;\le\; g_n$ that

$$\lim_n \int |f_n - f| \;=\; \int \lim_n |f_n - f| \;=\; 0 \, .$$

For the converse $\Longleftarrow$ we observe that

$$|\int |f| - \int |f_n|| \;\le\; \int ||f_n| - |f|| \;\le\; \int |f_n - f|$$

Thus the assertion follows immediately. ■

(4) (a) (10P) Let (a_n) be sequence of real numbers and $r \in \mathbb{R}$. Let us assume that every subsequence (a_{n_k}) has a further subsequence $(a_{n_{k_j}})_j$ such that

$$\lim_j a_{n_{k_j}} \;=\; r$$

Show that $\lim a_n \;=\; r$.

Solution: Assume not. Then there exists an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exists and $m > n$ with $|a_m - r| > \varepsilon$. Inductively we construct $m_1 > 1$ such that $|a_{m_1} - r| > \varepsilon$ and $m_2 > m_1$ such that $|a_{m_2} - r| > \varepsilon$,... This means m_k is an increasing subsequence such that

$$|a_{m_k} - r| > \varepsilon \ .$$

Obviously (a_{m_k}) does not permit a subseqeunce converging to r. ■

(b) (20P) Let g be a positive integrable function and (f_n) be sequence of measurable functions such that $|f_n| \leq g$ holds everywhere. Assume that f_n converges to f in measure. Show that

$$\lim_n \int f_n d\mu \ = \ \int f d\mu \ .$$

Solution: It suffices to show that for every subsequence (f_{n_k}) we find a further subsequence such that

$$\lim_j \int f_{n_{k_j}} \ = \ \int f \ .$$

However, for every subsequence (n_k) we may find a further subsequence (n_{k_j}) such that

$$\mu(|f_{n_{k_j}} - f| > 2^{-j}) < 2^{-j} \ .$$

Then $f_{n_{k_j}}$ converges to f a.e. The dominated convergence theorem implies the assertion. ■

Solutions for the practice problems

1) ('Easy question') Let $r_k > 0$ and $E_k \in \Sigma$ disjoint sets with finite measure. We assume that

$$\sum_k r_k \mu(E_k) \; < \; \infty$$

Show that

$$f \; = \; \sum_k r_k 1_{E_k}$$

is integrable and satisfies

$$I(f) \; = \; \sum_k r_k \mu(E_k) \; .$$

Hint: Use Fatou and dominated convergence theorem.

Proof: By Beppo Levi

$$\int \sum_k r_k 1_{E_k} = \int \sup_n \sum_{k \le n} r_k 1_{E_k} \; \le \; \sup_n \int \sum_{k \le n} r_k 1_{E_k} \; = \; \sup_n \sum_k r_k \mu(E_k)$$
$$= \sum_k r_k \mu(E_k) \; .$$

Hence $f = \sum_k r_k 1_{E_k}$ is integrable and we may apply the DCT for $f_n = \sum_{k \le n} r_k 1_{E_k}$ and $f = \lim_n f_n$ and majorant f.

Let f be an integrable function on $\mathbb{R}$ and g be a bounded measurable function. Show that

$$\lim_{t \to 0} \int |g(x)(f(x+t) - f(x))| \; = \; 0 \; .$$

Hint: First show this for f continuous and $f(x) = 0$ for $|x| \ge n$. Now use the fact that every integrable function f can be approximated by a continuous function h such that $\int |f - h| < \frac{\varepsilon}{3}$.

Proof: Let us assume that f continuous and $f(x) = 0$ for $|x| \ge n$. Let (t_k) be an arbitrary sequence converging to 0 such that $|t_k| \le 1$. Then

$$\lim_k |g(x)(f(x + t_k) - f(x))| \; = \; 0$$

for all $|x| in \mathbb{R}$. Since $f : [-n, n] \to \mathbb{R}$ is continuous it is also bounded. Let us define $C = \sup |f|$ and

$$h \; = \; \sup |g| 2C 1_{[-(n+1), n+1]} \; .$$

Given $x \in \mathbb{R}$ and $0 \le |t| \le 1$ we observe that

$$|g(x)(f(x+t) - f(x))| \; \le \; |g(x)||f(x+t)| + |f(x)| \; \le \; h(x)$$

because $x+t \in [-(n+1),(n+1)]$. Thus the dominated convergence theorem yields

$$\lim_k \int |g(x)(f(x+t_k)-f(x))| \;=\; \int \lim_k |g(x)(f(x+t_k)-f(x))| \;=\; 0\,.$$

Since (t_k) is arbitrary we deduce

$$\lim_{t\to 0} \int |g(x)(f(x+t)-f(x))|\,.$$

Now, let f be an arbitrary function and h be a continuous function with finite support such that $\int |f-h| < \frac{\varepsilon}{3\sup|g|}$ (see the notes on Lusin's theorem now on the web). We choose t_0 such that $|t|<t_0$ implies

$$\int |g(x)(h(x+t)-h(x))| < \frac{\varepsilon}{3}\,.$$

Then we get

$$\begin{aligned}
&\int |g(x)(f(x+t)-f(x))| \\
&\le \int |g(x)(f(x+t)-h(x+t))| + \int |g(x)(h(x+t)-h(x))| \int |g(x)(h(x)-f(x))| \\
&< \sup|g| \int |f(x+t)-h(x+t)| + \frac{\varepsilon}{3} + \sup|g| \int |(h(x)-f(x))| \\
&< \sup|g| \int |f(x+t)-h(x+t)| + \frac{2}{3}\varepsilon < \varepsilon\,.
\end{aligned}$$

In the last line we used

$$\int u(x+t)dm(x) \;=\; \int u(x)dm(x) \tag{0.2}$$

for arbitrary integrable functions. This equality (0.2) is proved similarly. First we observe that it is true for step functions (by the translation invariance of the Lebesgue measure (very easy for step functions)). Thus the linear map

$$T_t(f)(x) \;=\; f(x+t)$$

is defined on a dense subset L_1 with values in L_1 and satisfies

$$\|T_t(f)\|_1 \;=\; \|f\|_1$$

on a dense set. The unique extension principle allows us to extend T_t to a continuous linear map $\tilde{T}_t$ on $L_1(\mathbb{R})$ still satisfying

$$\|T_t(f)\|_1 \;=\; \|f\|_1\,.$$

We also have a Lipschitz map $I : L_1(\mathbb{R}) \to R$ given by

$$I([f]) \;=\; \int f(x)dm(x)\;.$$

Note that now for sequence (h_n) converging to f we have

$$I(\tilde{T}_t([f])) \;=\; \lim_n I([T_t(h_n)]) \;=\; \lim_n I([h_n]) \;=\; I([f]) \;=\; \int f(x)dm(x)\;.$$

Funny enough, it requires extra work to conclude that for $g \in \tilde{T}_t([f])$ we have $g = T_t(f)$ almost everywhere. Indeed, let (h_n) be a sequence of step functions which converges to f such that $\|f-h_n\| \leq 4^{-n}$. Then we may assume that (h_n) converges to f almost everywhere. Let F be a set of measure 0 such that $f(x) = \lim_n h_n(x)$ for all $x \in F^2$. By the definition of the outer measure we know that $F-t = \{x+t : x \in F\}$ has also measure 0. Let $x \in F^c \cap (F-t)^c$. Then $x+t \notin F$ and hence

$$\lim_n h_n(x) \;=\; f(x)$$

and

$$\lim_n h_n(x+t) \;=\; f(x+t)\;.$$

Thus $T_t(h_n)$ converges to $T_t(f)$ almost every where. Finally $\|T_t(h_n) - \tilde{T}_t(f)\|_1 = \|h_n - f\|_1 \leq 4^{-n}$ guarantees that $T_t(h_n)$ converges to g almost everywhere. Thus $g = T_t(f)$ holds almost everywhere and hence

$$\int f(x+t)dm(x) \;=\; \int_{(F\cap F-t)^c} f(x+t)dm(x) \;=\; \int g(x)dm(x) \;=\; \int f(x)dm(x)\;.$$

Don't look at the notes and show

(1) For a σ-finite measure space and a positive function f with $\int |f|^2 < \infty$ you can find an increasing sequence of simple functions $h_n \leq f$ such that

$$\int (|f|^2 - |h_n|^2) \;\leq\; 4^{-n}\;.$$

Conclude that $\mu(f^2 - h_n^2 > 2^{-n}) < 2^{-n}$. Thus h_n^2 converges to f^2 and henceforth h_n converges to f a.e. Use this to show

$$\lim_n \int |f-h_n|^2 \;=\; 0\;.$$

(2) Look at the web for the notes on the Lusin theorem and its consequences. Show that for every function $f : [-m,m] \to \mathbb{R}$ with $\int |f|^2 dm < \infty$ and $\varepsilon > 0$ there exists a continuous function g such that

$$\int |f-g|^2 dm < \varepsilon\;.$$

Solution: See notes.

2) Let (g_n) be a sequence positive integrable functions and (f_n) and integrable sequence such that $|f_n| \leq g_n$. We assume that f_n converges to f, g_n converges to g and

$$\lim_n \int g_n \;=\; \int g$$

Show that

$$\lim_n \int f_n \;=\; \int f \;.$$

(Remark: After the fact the argument can easily modified to the situation where a.e. is added in all the relevant places.)

Proof: Define $h_n \;=\; f_n + g_n$ which is positive. By Fatou

$$\int f+g \;=\; \int \liminf_n h_n \;\leq\; \liminf_n \int f_n + \int g_n \;=\; \liminf_n \int f_n + \int g \;,$$

because $\lim_n \int g_n = \int g$. Subtracting $\int g$ yields

$$\int f \;\leq\; \liminf_n \int -f_n \;.$$

Apply the same for $k_n = -f_n + g_n$ and we get

$$\int -f \;\leq\; \liminf_n \int -f_n \;.$$

Thus

$$\limsup_n \int f_n \;\leq\; \int f \;\leq\; \liminf_n \int f_n \;.$$

Thats it. ■

3) We will now discuss the metric associated to 'convergence in measure'. Let L_0 be the set of equivalence classes of measurable functions satisfying $\lim_{\alpha\to\infty} \mu(|f| > \lambda) = 0$.

(1) Show that

$$d([f],[g]) \;=\; \inf\{\varepsilon \;:\; \mu(|f-g| > \varepsilon) < \varepsilon\}$$

satisfies the triangle inequality.

Proof: Let h be a further function and $d([f],[h]) < \varepsilon$, $d([f],[h]) < \delta$ then

$$\mu(|f-h| > \varepsilon) < \varepsilon \quad \text{and} \quad \mu(|h-g| > \delta) < \delta \;.$$

Note that

$$\begin{aligned}\{\omega : |f(\omega) - g(\omega)| > \varepsilon + \delta\} &\subset \{\omega : |f(\omega) - h(\omega)| > \varepsilon \text{ or } |h(\omega) - g(\omega)| > \delta\} \\ &\subset \{\omega : |f(\omega) - h(\omega)| > \varepsilon\} \cup \{\omega : |h(\omega) - g(\omega)| > \delta\}\end{aligned}$$

because $|f(\omega)-h(\omega)| \leq \varepsilon$ and $|h(\omega)-g(\omega)| \leq \delta$ implies $|f(\omega)-g(\omega)| \leq \varepsilon+\delta$. Thus we have

$$\mu(|f-g| > \varepsilon+\delta) < \varepsilon+\delta \, .$$

Hence

$$d([f],[g]) \;\leq\; \varepsilon+\delta \, .$$

Taking the infimum yields the assertion. ■

(2) Show that if $([f_n])$ is Cauchy with respect to d, then there exists a subsequence $([f_{n_k}])$ such that

$$\mu(|f_{n_{k+1}} - f_{n_k}| > 2^{-k}) < 2^{-k} \, . \tag{0.3}$$

In this case (f_{n_k}) converges a.e.

Proof: We can always pass to a subsequence such that

$$d([f_{n_{k+1}}],[f_{n_k}]) < 2^{-k} \, .$$

Thus (0.3) holds. We follow the standard trick

$$E_j \;=\; \{\omega \;:\; |f_{n_{j+1}} - f_{n_j}| > 2^{-j}\}$$

and

$$F_k \;=\; \bigcup_{j\geq k} E_j$$

Then $\mu(F_k) \leq 2^{1-k}$ and hence $F = \bigcup_k F_k$ has measure 0. For $\omega \in F^c$ we can find k such that for all $j \geq k$

$$|f_{n_{j+1}}(\omega) - f_{n_j}(\omega)| \leq 2^{-j} \, .$$

Thus $f(\omega) \;=\; \lim_j f_{n_j}(\omega)$ exists on F^c. ■

(3) Show that (L_0, d) is a complete metric space.

Proof: It suffices to show that for every sequence $([f_n])$ satisfying $d([f_{n+1}],[f_n]) < 2^{-n}$ has a limit. By the argument above, we know that (f_n) converges a.e. for a limit f. Moreover, we use

$$F_n = \bigcup_{j\geq n} \{\omega : |f_{j+1}(\omega) - f_j(\omega)| > 2^{-j}\} \, .$$

Then $\mu(F_n) \leq 2^{1-n}$. For $\omega \in F_n^c$ we known that $f(\omega) \;=\; \lim_j f_j(\omega)$ converges and

$$|f(\omega) - f_n(\omega)| \;=\; |\sum_{j=n}^{\infty} f_{j+1}(\omega) - f_j(\omega)| \;\leq\; \sum_{j=n}^{\infty} |f_{j+1}(\omega) - f_j(\omega)| \;\leq\; 2^{1-n} \, .$$

Let $\varepsilon > 0$ then

$$\mu(|f - f_n| > (1+\varepsilon)2^{1-n}) \ \le\ 2^{1-n} < (1+\varepsilon)2^{1-n} \, .$$

This shows that

$$d([f_n],[f]) \ \le\ 2^{1-n} \, .$$

That's enough. ■

4) Let $\alpha > 0$. On the space vector space V of finite sequences

$$V \ = \ \{(a_n) \ : \ \exists_{n_0} \forall_{n > n_0} a_n \ = \ 0\}$$

we define the norm

$$\|(a_n)\| \ = \ \sum_n e^{\alpha n} |a_n|$$

Show that every continuous linear functional $\phi : (V, \| \ \|) \to \mathbb{R}$ is given by a sequence (x_n) and

$$\phi_{(x_n)}((a_n)) \ = \ \sum_n a_n x_n$$

and

$$\|\phi_{(x_n)}\| \ = \ \sup_n e^{-\alpha n} |x_n| \, .$$

Remark: For ODE the modified norms on $C(\mathbb{R})$

$$\|f\| \ = \ \sup_t e^{-\alpha |t|} |f(t)|$$

are important. Above you see a discrete analogue of this norm.

Proof: Let $\phi : V \to \mathbb{R}$ be a continuous linear functional of $\|\phi\| \le 1$. Then we may define

$$x_n \ = \ \phi(e_n) \, .$$

Here $e_n = (0 \cdots 0, \underbrace{\quad 1 \quad}_{n\text{-the position}}, 0 \cdots)$ is the n-th unit vector. Let $\varepsilon_n = 1$ of $x_n > 0$ and -1 else. Consider $a = e^{-\alpha n} \varepsilon_n e_n$. Then $\|a\| \le 1$ and hence

$$e^{-\alpha n} |x_n| \ = \ |\phi(a)| \ \le \ \|\phi\| \|a\| \ \le \ \|\phi\| \, .$$

Taking the supremum over $n \in \mathbb{N}$ yields the assertion. For the converse we assume

$$\phi_{(x_n)} \ = \ \sum_n x_n a_n$$

and $\sup_n e^{\alpha n} |x_n| \le 1$. Note that this sum is convergent because only finitely many terms are non zero. Thus we get

$$|\phi_{(x_n)}| = |\sum_n x_n a_n| \ = \ |\sum_n e^{-\alpha n} x_n e^{\alpha n} a_n|$$

$$\leq (\sup_n e^{-\alpha n}|x_n|) \sum_n e^{\alpha n}|a_n| \ = \ (\sup_n e^{-\alpha n}|x_n|)\|(a_n)\| \, .$$

Thus we have characterized exactly the continuous functionals of norm ≤ 1. This is enough. ■

Practice problems for the final

Let $1 \le p < \infty$ and f a measurable bounded function such that $f \in L_p$ show that for all $p < q < \infty$ we have $f \in L_q$ and

$$\lim_{q \to \infty} \|f\|_q \;=\; \|f\|_\infty \,.$$

Solution: Let $p < q < \infty$ and $c = \|f\|_\infty$. Note that $|f(x)| \le c$ holds a.e.

$$\int |f(x)|^q dm \le \int |f(x)|^p |f(x)|^{q-p} dm \;\le\; c^{q-p} \int |f(x)|^p dm \,.$$

Thus $f \in L_q(\mathbb{R})$. For the second part, we consider a natural number $m > c$. Let $n \in \mathbb{N}$ and define the simple function

$$h_l \;=\; \sum_{k=0}^{nm} \frac{k}{n}\, 1_{\frac{k}{n} < |f| \le \frac{k+1}{n}}$$

and

$$h^u \;=\; \sum_{k=0}^{nm} \frac{k+1}{n}\, 1_{\frac{k}{n} < |f| \le \frac{k+1}{n}} \,.$$

Note that $h_l \le |f| \le h^u$ and $f \in L_q$ implies that with Chebychev that $m(\frac{k}{n} < |f| \le \frac{k+1}{n}) < \infty$. By a previous hw problem we get for $h \in \{h_l, h^u\}$ and $a_k = k/n$ or $k + 1/n$ that

$$\lim_{p \to \infty} \|h\|_p \;=\; \lim_{k} (\sum_k a_k^p m(\frac{k}{n} < |f| \le \frac{k+1}{n}))^{\frac{1}{p}} \;=\; \sup_{k : m(\frac{k}{n} < |f| \le \frac{k+1}{n}) \neq 0} a_k \,.$$

Let k_c be such that $k_c < nc \le k_c + 1$. Thus we get

$$\lim_p \|h^u\|_p \;=\; k_c + 1/n$$

and

$$\lim_p \|h^u\|_p \;=\; k_c$$

This yields

$$\limsup_p \|f\|_p \;\le\; \lim_p \|h^u\|_p \;\le\; k_c + 1/n \;\le\; c + \frac{1}{n}$$

and

$$\liminf_p \|f\|_p \;\ge\; \lim_p \|h_l\|_p \;\ge\; k_c/n \;\ge\; c - \frac{1}{n} \,.$$

Letting $n \to \infty$ we deduce the assertion. ■

Show directly that for $1 \le p < \infty$ we have

$$\ell_p^* \;=\; \ell_{p'}$$

and that $\ell_\infty^* \neq \ell_1$.

Solution: Since we proved Holder's inequality for arbitrary measure space, we know that

$$|\sum_n a_nb_n| \ \le \ (\sum_n |a_n|^p)^{1/p}(\sum_n |b_n|^q)^{1/q}$$

whenever $1/p+1/q = 1$, i.e. $q = p'$. Thus the mapping $u : \ell_{p'} \to \ell_p^*$ defined by $u((b_n))((a_n)) \ = \ \sum_n a_nb_n$ is satisfies

$$\|u((b_n))\|_{\ell_p^*} \ \le \ (\sum_n |b_n|^q)^{1/q} \ .$$

We will now show that u is surjective. Indeed, let $\phi : \ell_p \to \mathbb{R}$ be a linear continuous map such that

$$|\phi(a_n)| \ \le \ (\sum_n |a_n|^p)^{1/p}$$

We define

$$b_n \ = \ \phi(e_n)$$

where e_n is the n-unit vector. Let $m \in \mathbb{N}$. From the equality consideration for the Holder inequality, we deduce

$$(\sum_{n\le m} |b_n|^q)^{1/m} \ \le \ \sup_{\sum_{n\le m} |a_n|^p \le 1} \|\sum_{n\le m} a_nb_n\| \ \le \ \|\phi\| \ .$$

Taking the sup over m we get

$$(\sum_n |b_n|^q)^{1/q} \ \le \ \|\phi\| \ .$$

Since $1 \le p < \infty$, we know that simple functions are dense. However, simple function here correspond to finite sequences. By the unique extension principle we deduce that the sequence (b_n) defined above satisfies $u(b_n) = \phi$ and

$$(\sum_n |b_n|^q)^{1/q} \ \le \ \|u((b_n))\|_{\ell_p^*} \ .$$

I will assume some knowledge in logic for proving $\ell_\infty^* \neq \ell_1$. (The argument breaks down because simple function=finite sequences are no longer dense). Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Then we define

$$\phi((a_n)) \ = \ \lim_{n,\mathcal{U}} a_n$$

One can show that for every compact set and every ultrafilter $\mathcal{U}$ the limit with respect to $\mathcal{U}$ exists. Here we may consider $(a_n) \subset [-c, c]$ and the ask $A_\{n \in \mathbb{N} :$

$-c \leq a_n \leq 0\} \in \mathcal{U}$ or $A_-^c \in \mathcal{U}$. The we split [-c,0] and (0,c] in two intervals and continue. It then easily follows that ϕ is well-define and satisfies

$$|\phi((a_n)| \;\leq\; \sup_n |a_n| \,.$$

However, ϕ does not come from an element in ℓ_1. Indeed, for every $(b_n) \in \ell_1$ and every infinite set A we have

$$c_k = u((b_n))(1_{A\cap[k,\infty]}) \;=\; \sum_{n\in A, n\geq k} b_k$$

and $\sum_k |b_k|$ finite implies $\lim_k c_k \;=\; 0$. However, let $A \in \mathcal{U}$ be an infinite subset. Then

$$\phi(1_{A\cap[k,\infty]}) = 1$$

holds for all k. ■

Problem 7a) and problem 7b) on page 104.

7a) If f is a monotone increasing function, then

$$\lim_{s\uparrow t} f(s) \;=\; \sup_{s<t} f(s)$$

and

$$\lim_{s\downarrow t} f(s) \;=\; \inf_{s>t} f(s) \,.$$

Thus these limits exist. A similar argument applies for monotone deceasing function. Thus for a function of bounded variation g we deduce the result by writing $g = f_1 - f_2$ with f_i increasing.

Consider again f monotone increasing on $[a,b]$. Let A be the set of continuity points. For fixed $n \in \mathbb{N}$ we consider

$$A_n \;=\; \{t \in (a,b) : \lim_{s\uparrow t} f(s) + \frac{1}{n} \;\leq\; \lim_{s\downarrow t} f(s)\}$$

Let $t_1, ..., t_m$ be m distinct elements. We may assume $a \leq t_1 < t_2 < \cdots t_m \leq b$. We choose points $a < s_1 < t_1 < s_2 < t_2 < s_3 < \cdots < t_m < s_m < b$. Then

$$f(b)-f(a) \;=\; f(b)-f(s_m)+f(s_m)-f(s_{m-1})+\cdots+f(s_2)-f(s_1)+f(s_1)-f(a) \;\geq\; m\frac{1}{n} \,.$$

Thus A_n has at most $n(f(b) - f(a))$ many elements. Since $\bigcup_n A_n$ is the collection of all discontinuity points in (a,b) we are done. ■

7b) Let $(r_n) \subset [0,1]$ be an enumeration of the rational points and (a_n) be a positive numbers sequence such that $\sum\limits_n a_n \;=\; 1$. We define

$$f \;=\; \sum_n a_n 1_{[r_n,1]}$$

The f is obviously monotone and has jumps at all points r_n, i.e. $\lim_{t\to r_n, t<r_n} f(t) + a_n = \lim_{t\to r_n} f(t)$. Now, let t be an irrational point. let $\varepsilon > 0$ and n_0 such that $\sum\limits_{n>n_0} a_n < \varepsilon$. Let $\delta = \min_{j=1,\dots,n_0} |t - r_j|$. For every $|t-s| < \delta$ we have

$$|f(t) - f(s)| \;\le\; \sum_{n>n_0} |a_n| < \varepsilon\ .$$

That's it. ■

Problem 10a) and problem 10b) on page 104.

10a) $g(x) = x^2 \cos(x^{-2})$. Let $n \in \mathbb{N}$ and define s_{n-j} such that $1/s_{n-j}^2 = \pi/2 + \pi j$ for $j = 0, ..., n$. Then we have

$$\begin{aligned}\sum_{j=0}^{n-1} |g(s_j) - g(s_{j+1})| &= \sum_{j=0}^{n-1} |s_{n-j}^2 + s_{n-j-1}^2| \\ &= \sum_{j=1}^{n} |s_j^2 + s_{j-1}^2| \\ &\ge \sum_{j=1}^{n} 1/(\pi/2 + \pi j) \;\ge\; \frac{1}{\pi}\sum_{j=1}^{n} 1/j\ .\end{aligned}$$

Since $\sum_j 1/j \;=\; \infty$ we deduce the assertion.

For b) and $g(x) = x^2 \sin(1/x)$ we note that it suffices to show that g' is in L_1. Except for 0 we have $g'(x) \;=\; 2x\sin(1/x) + x^2\cos(1/x)(-x^{-2}) \;=\; 2x\sin(1/x) - \cos(1/x)$. Thus g' is almost everywhere bounded and thus in L_1. Hence g is of bounded variation.

Problem 16) p111)-If time permits I will explain this problem Friday in class. No I didn't-but here is the solution.

a) Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Then we have

$$f(a) + \int_a^x f' dm \;\le\; f(x)$$

for every $x \in [a, b]$. Thus we may define $g(x) \;=\; f(a) + \int_0^x f' dm$ and $h(x) \;=\; f(x) - g(x)$. By the fundamental theorem (Lebesque differentiation theorem) we have $h'(x) \;=\; f'(x) - g'(x) \;=\; 0$ almost everywhere. Moreover, let $y < x$ then

$$h(x) - h(y) \;=\; f(y) - f(x) - \int_y^x f' dm \ge 0$$

be the differentiation theorem for monotone functions. Thus h is monotone and $h' = 0$ a.e.-i.e. h' is singular.

In b) and c) we prove the following

Lemma 0.1. *Let f be a monotone function. f is singular if and only if for every $a \leq x \leq b$ and $\varepsilon > 0$ and $\delta > 0$ there exists non-overlapping intervals $[y_j, y_j + d_j]$ such that*

$$\sum_j d_j < \varepsilon \quad \textit{and} \quad f(x) - f(a) < \sum_j (f(y_j + d_j) - f(y_j)) + \delta \, .$$

Proof. "$\Rightarrow$": Let f be singular. Let $\varepsilon > 0$ and $E = \{x : f'(x) = 0\} \cap [a+\varepsilon, b-\varepsilon]$. We may find an open subset 0 of (a, b) such that $E \subset O$ and $m(O) < m(E) + \varepsilon$. For every $x \in E$ and $\gamma > 0$ we may find $0 < h < \gamma$ such that

$$f(x+h) - f(x) < \gamma h$$

By the Vitali coverging lemma we obtain non overlapping intervals $([x_k, x_k + h_k])_{k=1,\ldots,m}$ such that

$$\sum_{k=1}^{m} h_k > m(E \cap [a+\varepsilon, b-\varepsilon]) - \varepsilon \;\geq\; b - a - 3\varepsilon \, .$$

Without loss of generality we may assume $a + \varepsilon < x_1 < x_1 + h_1 < x_2 < x_2 + h_2 < \ldots < x_m < x_m + h_m \leq b$. We define $y_0 = a$ and $d_1 = x_1 - y_1$, $y_j = x_j + h_j$ and $d_j = x_{j+1} - y_j$. Finally $d_m = b - y_m$. Then

$$b - a \;=\; \sum_{k=1}^{m} d_k + \sum_{j=0}^{m} d_j \;\geq\; (b-a) - 3\varepsilon + \sum_{j=0}^{m} d_j \, .$$

This yields $\sum_j d_j < 3\varepsilon$. On the other hand

$$\begin{aligned} f(b) - f(a) &= \sum_k (f(x_k + h_k) - f(x_k)) + \sum_j f(y_j + d_j) - f(y_j)) \\ &\leq \sum_k \gamma h_k + \sum_j f(y_j + d_j) - f(y_j)) \\ &\leq \gamma(b-a) + \sum_j f(y_j + d_j) - f(y_j)) \, . \end{aligned}$$

This is exactly what we want to prove for $x = b$. However, since f' also holds on $[a, x]$ we are done.

"$\Leftarrow$": Let $f = g + h$ such that h is singular and g is absolutely continuous and $g(a) = f(a)$. Let $a \leq x \leq b$ we want to show $g(x) = g(a)$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\sum_k h_k < \delta \Rightarrow \sum_j (g(x_k + h_k) - g(x_k)) < \varepsilon \, .$$

(g is absolutely continuous). By assumption we find non-overlapping interval $[y_j, y_j + d_j]$ such that $\sum_j d_j < \delta$ and

$$f(x) - f(a) < \sum_j [f(y_j + d_j) - f(y_j)] + \varepsilon \, .$$

Let x_k, h_k such that $\bigcup_k [x_k, x_k + h_k] \cup \bigcup_j [y_j, y_j + h_j] = [a, b]$ and the two unions only overlap in the endpoints. Then we get

$$\begin{aligned} f(x) - f(a) &< \varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] + \sum_j [g(y_j + d_j) - g(y_j)] \\ & 2\varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] \\ & \leq 2\varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] + \sum_k [h(x_k + h_k) - h(x_k)] \\ & = \; 2\varepsilon + h(x) - h(a) \; = \; 2\varepsilon + h(x) \, . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we get $f(x) \; = \; h(x) + f(a)$. Thus f is singular. ■

PROPOSITION 0.2. *Let (f_n) positive singular monotone functions such that $f(x) = \sum_n f_n$ converges point-wise. Then f is singular.*

PROOF. Without loss of generatily we may assume $f_k(a) = 0$ for all $k \in \mathbb{N}$. Let $a \leq x \leq b$, $\varepsilon > 0$, $\delta > 0$. Let n_0 be such that $\sum\limits_{k=1}^{n_0} f_k(x) > f(x) - \delta$. Obviously, $(\sum_{k=1}^{n_0} f_k)'(x) = \sum_{k=1}^{n_0} f_k'(x) \; = \; 0$ holds a.e. Thus we find non-overlapping intervals $([y_j, y_j + d_j])$ such that $F_{n_0} = \sum\limits_{k=1}^{n_0} f_k$ satisfies $\sum_j d_j < \delta$ and

$$F_{n_0}(x) - \varepsilon < \sum_j [F_{n_0}(y_j + d_j) - F_{n_0}(y_j)] \, .$$

This implies by monotonicity of the f_n's and by point-wise convergent that

$$\begin{aligned} f(x) - 2\varepsilon = F_{n_0}(x) &< \sum_j [F_{n_0}(y_j + d_j) - F_{n_0}(y_j)] \\ &= \sum_j \sum_{n=1}^{n_0} [f_n(y_j + d_j) - f(y_j)] \\ &= \sum_j \sum_{n=1}^{\infty} [f_n(y_j + d_j) - f(y_j)] \\ &= \sum_j [f(y_j + d_j) - f(y_j)] \, . \end{aligned}$$

Thus f is also singular-i.e. a function which creates everything out of nothing. ■

e) Finally consider (r_n) an enumeration of the rationals and (a_n) strictly positive such that $\sum_n a_n < \infty$. Then

$$f \;=\; \sum_n a_n 1_{[r_n,1]}$$

is singular by our previous proposition. Moreover, f is strictly increasing because between two points $x < y$ we find $x < r_n < y$ and hence $f(y) - f(x) > a_n$. ■

Let $\mu << \nu$ be finite probability measures. Let f_1 and f_2 be Radon Nikodym derivatives such that

$$\mu(E) \;=\; \int\limits_E f_1 d\nu \quad \text{and} \quad \mu(E) \;=\; \int\limits_E f_2 d\nu$$

What can you say about f_1 and f_2. In which sense is the Radon-Nikodym derivative unique.

Solution: Consider $E_n = \{\omega \in \Omega : f_1(\omega) > f_2(\omega) + \frac{1}{n}\}$. Then

$$0 \;=\; \int_E (f_1 - f_2) d\nu \;\geq\; \frac{1}{n}\nu(E) \;\geq\; 0\,.$$

Thus $\nu(E_n) = 0$. This implies $f_1 = f_2$ holds ν almost everywhere. Hence the Radon-Nikodym derivative is uniquely determined up to set of measure 0 for ν. ■

Let $\Omega = \{1, ..., n\}$ and ν the counting measure $\nu(A) = |A|$. Let μ be an arbitrary measure calculate the Radon-Nikodym derivative.

Solution: We consider the positive numbers

$$\mu_i \;=\; \mu(\{i\})\,.$$

Define $f(i) = \mu_i$. Then

$$\mu(E) \;=\; \sum_{i \in E} \mu_i \;=\; \int_E f d\nu$$

holds for every set. ■

Final exam 540-December 04

Name:

In the first three problems extra credit is given if you prove some of the statements. You will need around 115 points ($\pm\varepsilon$ to be determined later) our of 125 points for an A.

(1) (15P) State the Baire category theorem and formulate an application in terms of uniform boundedness.

(2) (15P) Formulate the unique extension principle. Use it to show that that for $g \in C[0,1]$ the map $T : (C[0,1], \| \|_1) \to (C[0,1], \| \|_1)$ given by $T(f)(t) = f(t)g(t)$ extends to $L_1[0,1]$. Can you describe the extension?

(3) (15P) Give an example of a continuous not absolutely continuous function.

(4) (20P) Let $1 \leq p < r < q < \infty$ and f be a measurable function such that $f \in L_p(\mathbb{R})$ and $f \in L_q(\mathbb{R})$. Show that $f \in L_r(\mathbb{R})$. (Hint: Look at the set $\{x : |f(x)| \geq 1\}$).

(5) (30P) Let $(a_j)_{j=1}^n$ be a sequence of strictly positive real numbers and $x_1, ..., x_n \in [0, 1]$ be distinct real numbers. Consider the measure

$$\mu(A) \;=\; \sum_{j=1}^{n} a_j 1_A(x_j)\ .$$

(a) Is it true that $\mu << m$ (is absolutely continuous with respect to the Lebesgue measure). Give a proof for your answer.

(b) Let $(b_j)_{j=1}^n$ be a sequence of positive numbers and

$$\nu(A) \;=\; \sum_{j=1}^{n} b_j 1_A\ .$$

Find a Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. Can you find two?

(6) (30P) Let F be a function of bounded variation on $[a, b]$. Show that

$$\int_a^b |F'| dm \;\leq\; \|F\|_{BV} \,.$$

State explicitly the results you use in the proof. Give an example where strict equality holds (extra credit).