Introduction to real analysis -hw2

Due date: Monday, September 13

i) (30p) Show that

$$B = \{x \in \mathbb{R}^2 : d_{SNCF}(x,(0,0)) \leq 1\}$$

is closed bounded, but not compact (in $(\mathbb{R}^2, d_{SNCF})$). Let $u : B \to (\mathbb{R}^m, d_2)$ be an injective map such that

$$d_2(u(x),u(y)) \leq d_{SNCF}(x,y) .$$

Show that are sequences (x_n), $(y_n) \subset B$ such that

$$\lim_n \frac{d_2(u(x_n),u(y_n))}{d_{SNCF}(x_n,y_n)} = 0 .$$

(Hint: If this is not the case $\inf \frac{d_2(u(x),u(y))}{d_{SNCF}(x,y)} \geq c > 0$ (why?). However, this leads to a contradiction if you have done your job before correctly).

ii) (20P) Let $(r_n) \subset [0,1]$ be a strictly increasing sequence. Let $(\alpha_n) \subset [0,1]$ be such that $\sum_n \alpha_n$. We define

$$f_n(x) = \alpha_n 1_{[r_n,1]}(x) = \alpha_n \begin{cases} 1 & \text{if } x \geq r_n \\ 0 & \text{else} \end{cases} .$$

Show that for every $x \in [0,1]$

$$f(x) = \sum_n f_n(x)$$

is well-defined. Show that f is upper semi-continuous. Let $y \in \mathbb{R}^2$. Show that $f(x) = -d_{SCNF}(x,y)$ is upper semicontinous.

iii) (30p) Show that $C[0,1]$ with is not complete with respect to

$$d_1(f,g) = \int_0^1 |f(t) - g(t)| .$$

iv) (10p) Show that a closed subset of a compact set is compact. (If possible use the definition with open covers).

v) (10P). Show Lindelöf's theorem in $\mathbb{R}^m$: Every open subset of $\mathbb{R}^m$ is a countable union of open balls.

Introduction to real analysis -hw1

Due date: Monday, September 13

i) Show that

$$B \;=\; \{x \in \mathbb{R}^2 : d_{SNCF}(x,(0,0)) \le 1\}$$

is closed bounded, but not compact (in $(\mathbb{R}^2, d_{SNCF})$). Let $u : B \to (\mathbb{R}^m, d_2)$ such that

$$d_2(u(x),u(y)) \;\le\; d_{SNCF}(x,y) \;.$$

Show that are sequences (x_n), $(y_n) \subset B$ such that

$$\lim_n \frac{d_2(u(x_n),u(y_n))}{d_{SNCF}(x_n,y_n)} \;=\; 0 \;.$$

ii) Let $(r_n) \subset [0,1]$ be a strictly increasing sequence. Let $(\alpha_n) \subset [0,1]$ be such that $\sum_n \alpha_n$. We define

$$f_n(x) \;=\; \alpha_n 1_{[r_n,1]}(x) \;=\; \alpha_n \begin{cases} \alpha_n 1 & \text{if } x \ge r_n \\ 0 & \text{else} \end{cases} \;.$$

Show that for every $x \in [0,1]$

$$f(x) \;=\; \sum_n f_n(x)$$

is well-defined. Show that f is upper semi-continuous.

(1) Show that $C[0,1]$ with is not complete with respect to

$$d_1(f,g) \;=\; \int_0^1 |f(t)-g(t)| \;.$$

Introduction to real analysis -hw3

Due date: Wednesday, September 20

(1) (20P) Show that $[0,1]$ is compact by verifying the definition.

(2) i) (10P) Given an example of a continuous function and a closed set C such that $f(C)$ is not closed.
   ii) (10P) Show that image of a compact set under a continuous map is compact.
   ii) (15P) Show that image of a relatively compact set under a continuous map is relatively compact.
   iv) (10P) Let X be a compact metric space and $f : X \to Y$ be continuous and bijective. Show that the inverse function f^{-1} is continuous.

(3) (30P) Let us consider the set

$$Lip_c \;=\; \{f \in C[0,1] \;:\; |f(x) - f(y)| \le c|x-y|\} \; .$$

Show that Lip_c is not relatively compact but

$$F \;=\; Lip_c \cap B(0,1) \;=\; \{f \in Lip_c \;:\; \sup_{0\le x\le 1} |f(x)| \le 1\}$$

is relatively compact in $C[0,1]$. As an application, we consider $I : C[0,1] \to C[0,1]$ given by

$$I(f)(t) \;=\; \int_0^t f(s)ds \; .$$

Show that

$$\{I(f) \;:\; \sup_{0\le x\le 1} |f(x)| \le 1\}$$

is relatively compact in $C[0,1]$.

(4) (20P) No 36a) on page=161: A point x in a metric space is called isolated if the set $\{x\}$ is open. Show that a complete metric space without isolated points has an uncountable number of points.

Introduction to real analysis -hw4

Due date: Monday, September 27

(1) (10P)

(a) Let F is nowhere dense if and only if every nonempty open set $O \cap F^c$ contains an open ball.

(b) Show that the countable union of meager sets is meager.

(2) (25P) We want to show that for an upper semicontinuous function $f : X \to \mathbb{R}$ on a complete metric space the set of continuity points is dense.

(a) Let $a < b$ in $\mathbb{R}$. Show that

$$A_{a,b} \;=\; \{x \in X : f(x) \geq b \text{ and } \exists_{(x_j),\lim_j x_j = x} \lim_j f(x_j) \leq a\}$$

is closed.

(b) Show that the sets $A_{[a,b]}$ have nonempty interior.

(c) For $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ we define $B_{k,m} \;=\; A^c_{[\frac{k-1}{m},\frac{k}{m}]}$. Show that for $x \in \bigcap_{k,m} B_{k,m}$ f is continuous at x. Show that for complete X the set of continuity points is dense.

(3) (20P) Let $f : X \to Y$ be a function. Define

$$\omega_\delta(f)(x) \;=\; \sup\{d'(f(y), f(z) \;:\; d(x,y) < \delta \text{ and } d(x,z) < \delta\}\;.$$

Let $\varepsilon > 0$. Show that $\{x : \omega_\delta(f)(x) < \varepsilon\}$ is open. Conclude that the set $C(f)$ of continuity points can be written as a countable intersection of open sets.

(4) (25) Let Y be a separable metric space and X a complete metric space. Let $f : X \to Y$ be a function such that $f^{-1}(\bar{B}(y,\varepsilon))$ is closed for every $y \in Y$ and $\varepsilon > 0$. Then the sets of continuity points is dense in X.

(5) (20P) Let X be a set with an uncountably many elements (you might work with $\mathbb{R}$). Show that

$$A \;=\; \{S \subset X \;:\; S \text{ or } |X \setminus S| \text{ is countable}\}$$

is a σ-algebra.

Introduction to real analysis -hw5

Due date: Monday, October 4

(1) (15P) Show that for set $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and $\varepsilon > 0$ there exists a compact set $C \subset E$ such that $m(C) \leq m(E) < m(C) + \varepsilon$. Conclude that for measurable E we have $m(E \setminus C) < \varepsilon$.

(2) (20P) Let μ be a σ-additive measure on $\mathcal{B}$ and (E_j) events in B,

(a) Show that if $E_1 \supset E_2 \cdots$ and $m(E_1) < \infty$, then
$$\mu(\bigcap_j E_j) \;=\; \lim_j \mu(E_j)\,.$$
Show that the assumption $\mu(E_1) < \infty$ is really needed.

(b) Show that if $E_1 \subset E_2 \cdots$, then
$$\mu(\bigcup_j E_j) \;=\; \lim_j \mu(E_j)\,.$$

(3) (30P) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function such that $f(x) = \lim_{y<x, y\to x} f(y)$. (That means f is continuous from the left.) We define
$$m_f([a,b)) \;=\; f(b) - f(a)\,.$$
Show that for every interval $I = [a,b]$ we have
$$f(b) - f(a) \;\leq\; \inf\{\sum_j m_f(I_j) \;:\; I \subset \bigcup_j I_j\}\,.$$
Here the infimum is taken over right open intervals $I_j = [a_j, b_j)$ (in principle we allow $a_j = -\infty$ and consider $(-\infty, b_j)$ as half open). Show that
$$f(b) - f(a) \;=\; \inf\{\sum_j m_f(I_j) \;:\; [a,b) \subset \bigcup_j I_j\}\,.$$

(4) (30P) We will need an estimate using Sterling's formula, namely
$$\lim_n 2^{-2n}\binom{2n}{n} \;=\; 0\,.$$
(You can amuse yourself in finding Sterling's formula and how to deduce that.)

(a) Let $X_n = \{-1,1\}^n$ and $\mu_n(A) = 2^{-n}|A|$ (where $|A|$ is the cardinality of A). Show that
$$\mu_n(\{(\varepsilon_1, ..., \varepsilon_n) : \sum_{i=1}^n \varepsilon_i = k\}) \;=\; 2^{-n}\binom{n}{\frac{n+k}{2}}\,.$$

Deduce from this that for even n

$$\mu_n(\{(\varepsilon_1, ..., \varepsilon_n) : |\sum_{i=1}^{n} \varepsilon_i| \leq k\}) \;=\; 2k2^{-n}\binom{n}{\frac{n}{2}} \,.$$

(b) On $X_\infty = \{-1,1\}^\infty$ we denote by μ the extension of then μ_n's (explained in class). Show that for every k

$$\mu(\{(\varepsilon_1, ...) \;:\; \sup_n |\sum_{j=1}^{n} \varepsilon_j| \;\leq\; k\}) \;=\; 0 \,.$$

(c) Let X_∞. Show that

$$\mu(\{(\varepsilon_1, ...) \;:\; \lim_n \sum_{j=1}^{n} \varepsilon_j \text{ exists }\}) \;=\; 0 \,.$$

(5) Real Analysis-Homework 6

Due date: Monday, October 18

(1) Let us denote by $p_k(t) = t^k$, $k \in \mathbb{N}_0$ the polynomials. Show that

$$\{a_0 p_0 + a_1 p_1 + a_2 p_2 \ : \ a_0, a_1, a_2 \in \mathbb{R}\}$$

is a closed subset of $C[0,1]$. (Hint: Show that for $R > 0$ the set

$$\{(a_0, a_1, a_2) : \sup_{0 \le t \le 1} |a_0 + t a_1 + t^2 a_2| \le R\}$$

is closed and bounded.)

(2) Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\phi(x_1, x_2) \ = \ (2x_1, \frac{1}{2} x_2) \ .$$

Show that for every Lebesgue measurable subset E of $\mathbb{R}^2$.

$$m(\phi^{-1}(E)) \ = \ m(E) \ .$$

(3) (a) Use an enumeration of the rational numbers in $[0,1]$ and construct an open set of measure $< \frac{1}{n}$ which contains the rational numbers.

(b) Construct a meager subset of $[0,1]$ of measure 1.

Real Analysis-Homework 7

Due date: Monday, October 25

(1) (25P) Let $P \subset [0,1]$ be the non measurable set constructed in class (as set of representatives of $x \sim y$ iff $x - y \in \mathbb{Q}$). Show that for every measurable subset $E \subset P$ we have $m(E) = 0$. (Royden: 3.15)

(2) (25P) Show that every set with $0 < m^*(A) < \infty$ contains a non measurable set (Royden 3.16).

(3) (25P) Show Proposition 24 on page 73 (Royden): Let E be a measurable set of finite measure and (f_n) a sequence of measurable functions which converges to a real valued function f a.e. Show that for every $\varepsilon > 0$ and $\delta > 0$, there exists a subset $A \subset E$ such that $m(A) < \delta$ and $n_0 \in nz$ such that for $n \geq n_0$ and $x \notin A$

$$|f_n(x) - f(x)| < \varepsilon \, .$$

(4) (25P) (Royden 3.30) Prove Egoroff's theorem: Let E be a measurable set of finite measure and (f_n) a sequence of measurable functions which converges to a real valued function f a.e. Show that on a set of large measure (f_n) converges uniformly to f.

Real Analysis-Homework 8

Due date: Monday, November 1

(1) (15P) Let $\Omega = \mathbb{R}$, $\Sigma = \{A : A \text{countable or } A^c \text{countable}\}$ and

$$\mu(A) \;=\; \begin{cases} \infty & A^c \text{ is countable} \\ 0 & A \text{ is countable} \end{cases} \;.$$

Show that μ is σ-additive. Consider $f = 1_{\mathbb{R}}$. Show that

$$I(f) \;=\; 0 \quad \mu(\{x \in \mathbb{R} : f(x) \geq \frac{1}{2}\}) \;=\; \infty\,.$$

(2) Let (Ω, Σ, μ) be a measure space (not necessarily σ-finite). We will now say that a function $f : \Omega \to [-\infty, \infty]$ is measurable in the strong sense if there exists a set $F \in \Sigma$ of measure 0 and a sequence (g_n) of simple functions such that

$$f(\omega) \;=\; \lim_n g_n(\omega)$$

holds for all $\omega \in F^c$.

(a) (15P) Let $f \geq 0$ and measurable in the strong sense. Show that for every $\lambda > 0$ the set $E_\lambda = \{\omega \in \Omega : f(\omega) \geq \lambda\}$ is σ-finite, i.e. there exists $G_n \in \Sigma$ with finite measure such that $E_\lambda \;=\; \bigcup_n G_n$. (Hint: use the functions $h_n = \inf_{m \geq n} g_n$.

(b) (10P) Let (Ω, Σ, μ) be a finite measure space. Show that every measurable function is measurable in the strong sense. (Hint: use the functions f_ε below).

(3) (20P) Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) < \infty$. Let $f \geq 0$ and $\varepsilon > 0$. Consider

$$f_\varepsilon \;=\; \sum_{k=0}^{\infty} (k\varepsilon) 1_{\{k\varepsilon \leq f < (k+1)\varepsilon\}}\,.$$

Show that

$$I(f) \;=\; \lim_{\varepsilon \to 0} I(f_\varepsilon)\,.$$

Conclude that

$$I(f) \;=\; \inf\{\sum_k r_k \mu(E_k) \;:\; f \;\leq\; \sum_k r_k 1_{E_k}\}\,.$$

(4) In this exercise we want to establish the link between areas and integrals. Let (Ω, Σ, μ) be a finite measure space ($\mu(\Omega) < \infty$). On $\tilde{\Omega} \;=\; [0, \infty) \times \Omega$ we consider the algebra A generated by the sets $[s, t) \times E$, $0 \leq s \leq t \leq \infty$, $E \in \Sigma$.

(a) (10P) Show that every element F in A can be written as

$$F = \bigcup_{k=1}^{m} G_k \times E_k$$

such that the E_k's are disjoint and $G_k \in A_{\mathbb{R}}$. we then define

$$\nu(F) = \sum_k m(G_k)\mu(E_k) .$$

It can be shown that for every $F \in A$ and for every disjoint union $F = \bigcup_j F_j$ of disjoint sets in A we have

$$\nu(F) = \sum_j \nu(F_j) .$$

Therefore we may extend ν to a σ-additive measure on the σ-algebras $\tilde{\Sigma}$ of measurable subsets (in the Caratheodory sense) of $[0,\infty] \times \Omega$.

(b) (5P) Let $E \in \Sigma$ be a set of measure 0. Show that

$$\nu([0,\infty) \times E) = 0 .$$

(c) (15P) Let $f : \Omega \to [0,\infty]$ be measurable such that $I(f) < \infty$. Show that the *graph of f*

$$G(f) = \{(r,\omega) : 0 \le r \le f(\omega)\}$$

has finite measure with respect to ν and satisfies $\nu(G(f)) \le I(f)$. (Hint for a simple function h we have $\nu(G(h)) = I(h)$.

(d) (10P) Let $f : \Omega \to [0,\infty]$ be measurable and assume

$$\nu(G(f)) < \infty .$$

Show that $I(f) \le \nu(G(f))$. (Hint: consider a simple function $0 \le h \le f$).

Real Analysis-Homework 9

Due date: Monday, November 15

(1) (a) (10P) Let (x_k) be a sequence of real numbers such that

$$f((\alpha_k)) \;=\; \sum_{k=1}^{n} x_k \alpha_k$$

exists for all $(\alpha_k) \subset \ell_2$. Show that $\sum_k |x_k|^2$ is finite.

(b) (20P) Let X be a Banach space and $(f_n) \subset X^*$ be a sequence of linear functionals such that

$$f(x) \;=\; \lim_n f_n(x)$$

converges for all $x \in X$. Show that f is continuous.

(2) (30P) Problem 18 in chapter 4 p=94 in Royden.

(3) (30P) Problem 19 in chapter 4 p=94 in Royden.

Real Analysis-Homework 10

Due date: Wednesday, December 1

(1) (a) Let $x_1, ..., x_n \in \mathbb{R}$. Show that

$$\sup_i |x_i| \;\le\; \left(\sum_{i=1,..,n} |x_i|^p \right)^{\frac{1}{p}} \;\le\; n^{\frac{1}{p}} \sup_{i=1}^{n} |x_i| \,.$$

(b) Let $f = \sum_i x_i 1_{E_i}$ be a simple function. Show that

$$\lim_p \|f\|_p \;=\; \|f\|_\infty \,.$$

(c) Let f be measurable function such that $[f] \in L_\infty$ show that

$$\lim_p \|f\|_p \;=\; \|f\|_\infty \,.$$

Hint: For a finite measure space you may use suitable simple function to approximate f from below and above. The general σ-finite case requires an additional approximation argument.

(2) (a) Let $V = \{(x_n) \;:\; \exists_{k\in\mathbb{N}} \forall_{n>k} x_n \;=\; 0\}$ the space of finite sequences. We use

$$\|(x_n)\|_\infty \;=\; \sup_n |x_n| \,.$$

on V. Show that

$$\phi((x_n)) \;=\; \sum_n x_n$$

is a linear map on V which is not continuous.

(b) Let $(V, \|\ \|)$ be a normed space and $\phi : V \to \mathbb{R}$ be a linear map. Show that ϕ is continuous if and only if

$$\{x \in V \;:\; \phi(x) = 0\}$$

is closed. (Hint: One implication is easy. For the other implication you may assume that there is a $x_0 \in V$ with $f(x_0) = 1$. Then $d = \inf\{\|x_0 - y\| \;:\; \phi(y) = 0\} > 0$ (why?). Use this to show that for arbitrary x with $f(x) = 1$ we have $\|x\| \ge d$ because $\phi(x - x_0) = 0$. Conclude).

Real Analysis-Homework 11

Due date: Monday, December 6

(1) Royden 5.12 (p=110)
(2) Royden 5.14 (p=111)
(3) Royden 5.15 (p=111)
(4) Royden 5.17 (p=111) (Assume in addition that g is monotone increasing).