CHAPTER 5

Integration in $\mathbb{R}$

1. Absolute continuous functions

A function $f : [a,b] \to \mathbb{R}$ is called of bounded variation if

$$\|f\|_{BV} \;=\; \sup\{\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \;:\; a = x_0 < x_1 < \cdots < x_n = b\}$$

is finite. We say that f is absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition $a = x_0 < x_1 < \cdots < x_n = b$ and every subset $J \subset \{1, ..., n\}$

$$\sum_{i \in J} |x_{i+1} - x_i| < \delta \Longrightarrow \sum_{i \in J} |f(x_{i+1}) - f(x_i)| < \varepsilon \,.$$

`mot` Lemma 1.1. *Let* $f \in L_1[a,b]$ *and* $F(t) = \int_a^t f(s)dm(s)$*. Then* F *is of bounded variation and absolutely continuous. Moreover,* $F(a) = 0$ *and* $\|F\|_{BV} \leq \int |f|$*.*

Proof. For a partition $a = x_0 < x_1 < \cdots < x_n = b$ and $J \subset \{1, .., n\}$ and $\varepsilon_i = \frac{F(x_{i+1} - F(x_i))}{|F(x_{i+1} - F(x_i))|}$ we have

$$\sum_{i \in J} |F(x_{i+1} - F(x_i))| = |\int (\sum_{i \in J} \varepsilon_i 1_{[x_i, x_{i+1}]}) f dm| \;\leq\; \int |f| 1_{\bigcup_{i \in J}[x_i, x_{i+1}]} dm \,.$$

Thus for $J = \{1, .., n\}$ we get

$$\sum_{i=0}^{n-1} |F(x_{i+1}) - F(x_i)| \;\leq\; \int |f| dm \,.$$

The absolute continuity follows from

$$\lim_{m(A) \to 0} \int_A |f| dm \;=\; 0 \,.$$

(see exam). ■

LEMMA 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a of bounded variation. Then f is the difference of two monotone functions f_1, f_2. If in addition f is absolutely continuous, then f_1 and f_2 may be assume absolutely continuous.*

$$\|f\|_{BV} \;=\; f_1(b) + f_2(b) - f(a) \,.$$

PROOF. For any partition π we define

$$\begin{aligned} p(f,\pi) &= \sum_{i=0}^{n-1} \max\{f(x_{i+1}) - f(x_i), 0\} \\ n(f,\pi) &= \sum_{i=0}^{n-1} \max\{-f(x_{i+1}) + f(x_i), 0\} \\ t(f,\pi) &= \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \,. \end{aligned}$$

Then

$$t(f,\pi) \;=\; p(f,\pi) + n(f,\pi)$$

and

$$f(b) - f(a) \;=\; \sum_{i=0}^{n} (f(x_{i+1}) - f(x_i)) \;=\; p(f,\pi) - n(f,\pi) \,.$$

This implies

$$f(b) - f(a) + n(f,\pi) \;=\; p(f,\pi) \,.$$

Now, we take the supremum over all partitions and still have

$$f(b) - f(a) + \sup_{\pi} n(f,\pi) \;=\; \sup_{\pi} p(f,\pi) \,.$$

Moreover,

$$\begin{aligned} \sup_{\pi} t(f,\pi) = \sup_{\pi}[p(f,\pi) + n(f,\pi)] \;&=\; \sup_{\pi}[2p(f,\pi) - (f(b) - f(a))] \\ = \sup_{\pi} p(f,\pi) + \sup_{\pi} p(f,\pi) - (f(b) - f(a)) \;&=\; \sup_{\pi} p(f,\pi) + \sup_{\pi} n(f,\pi) \,. \end{aligned}$$

For $a \le x \le b$ we define

$$g(x) \;=\; \sup_{\pi=\{a=x_0<\ldots<x_n=x\}} p(f,\pi)$$

and

$$h(x) \;=\; \sup_{\pi=\{a=x_0<\ldots<x_n=x\}} n(f,\pi) \,.$$

Then g and h are increasing functions and

$$f(x) - f(a) + h(x) \;=\; g(x) \,.$$

This yields

$$f(x) \;=\; g(x) - h(x) + f(a)\ .$$

Moreover,

$$\|f\|_{BV} \;=\; g(b) - f(a) + h(b)\ .$$

If in addition f is absolute continuous then it follows by the definition that $\sum_i |x_{i+1} - x_i| < \delta$ implies

$$\sup_{i \in J} \sum_{i \in J} |g(x_{i+1}) - g(x_i)| \;\le\; \sup_{\sum_j |y_{j+1} - y_j| < \delta} \sum_j \max\{f(y_{j+1}) - f(y_j), 0\}\ .$$

Thus we can work with same relation between ε and δ for g and h. ■

Theorem 1.3. *Let $F : [a, b] \to \mathbb{R}$ be an absolute continuous function of bounded variation. Then there exists a function $f \in L_1[a, b]$ such that*

$$F(t) \;=\; F(a) + \int_a^t f(s)ds$$

and $\|f\|_1 \;=\; \|F\|_{BV}$.

Proof. We may assume $F(a) = 0$. Let $F \;=\; F_1 - F_2$ such that F_1 and F_2 are positive

$$F_1(b) + F_2(b) \;=\; \|F\|_{BV}$$

and such that F_1, F_2 are absolutely continuous. We define the measure on $A_{\mathbb{R}}$.

$$\nu((s, t]) \;=\; F_1(t) - F(s)\ .$$

Using the absolute continuity it is not hard to check that

$$\nu((s, t]) \;=\; \sum_j \nu((s_j, t_j])$$

for every disjoint decomposition. Thus ν extends to a σ additive measure on the borel sets which is absolutely continuous with respect to the Lebesque measure. Thus ν extends to Lebesgue measurable set. By the Radon-Nikodym theorem we find a measurable function f_1 such that

$$F_1((s, t]) \;=\; \int_s^t f_1 dm\ .$$

Then

$$\int f_1 dm \;=\; F_1(b) - F_1(a) \;=\; F_1(b)\ .$$

We apply the same argument to F_2 and find a positive element f_2 such that

$$\int f_2 dm \;=\; F_2(b)\ .$$

Thus $f = f_1 - f_2$ satisfies the assertion by Lemma `mot` 1.1. ■