4. Banach spaces

DEFINITION 4.1. *A normed space is given by a vector space* V *(over* $K = \mathbb{R}$ *or* $K = \mathbb{C}$*) and a function* $\| \; \| : V \to [0, \infty)$ *satisfying the following conditions*

i) $\|x\| = 0 \Leftrightarrow x = 0$,

ii) $\|\lambda x\| = |\lambda| \|x\|$,

iii) $\|x + y\| \leq \|x\| + \|y\|$,

for all $x, y \in V$, $\lambda \in K$. *The associated metric on* $(V, \| \; \|)$ *is defined by*

$$d_{\| \; \|}(x, y) \;=\; \|x - y\| \,.$$

REMARK 4.2. $+ : V \times V \to V$ *given by* $+(x, y) = x + y$ *and* $\cdot : K \times V \to V$ *given by* $\cdot(\lambda, x) = \lambda x$ *are continuous. Moreover,* $\| \; \| : V \to [0, \infty)$ *is continuous.*

In the following we will mostly consider real vector spaces (because the name of our course is real analysis).

DEFINITION 4.3. *A Banach space is a normed vector space such that* $(V, d_{\| \; \|})$ *is complete.*

EXAMPLE 4.4.

(1) *On* $V = \mathbb{R}^n$ *we define*

$$\|x\|_p \;=\; \left(\sum_{i=1}^{n} \|x_i|^p \right)^{\frac{1}{p}}$$

and $\|x\|_\infty = \max_{i=1,\ldots,n} \|x_i|$. *Then* $(\mathbb{R}^n, \| \; \|_p)$ *is Banach space (see below for the triangle inequality).*

(2) $\ell_p = \{(x_n) \; : \; \sum_n |x_n|^p < \infty\}$ *is a Banach space with respect to*

$$\|(x_n)\|_p \;=\; \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} \,.$$

(3) *If* $\| \; \|$ *is a norm on* $\mathbb{R}^n$, *then* $(\mathbb{R}^n, \| \; \|)$ *is a Banach space.*

(4) $(C[0, 1], \| \; \|_1)$ *where*

$$\|f\|_1 \;=\; \int_0^1 |f(s)| ds$$

is a normed space, but not a Banach space.

PROPOSITION 4.5. *Let* X *be a normed space and* Y *be a Banach space. We define* $L(X, Y)$ *as the space of map* $T : X \to Y$ *which are linear, i.e.*

$$T(x + \lambda y) \;=\; T(x) + \lambda T(y) \,.$$

and continuous. The norm on $L(X,Y)$ *is given by*

$$\|T\|_{op} \;=\; \sup_{\|x\|\le 1} \|T(x)\| \;.$$

Then $L(X,Y)$ *is a Banach space.*

PROOF. Let us first show that a linear map $T : X \to Y$ is continuous iff $\|T\| < \infty$. Indeed, if $\|T\|$ is finite, then

$$\|T(x) - T(y)\| \;=\; \|T(x-y)\| \;\le\; \|T\|_{op}\|x-y\|$$

holds for all $x, y \in V$. Thus T is Lipschitz and thus continuous. For the converse, we assume that T is continuous. Then $T^{-1}(B(0,1))$ is open and henceforth contains $B(0,\varepsilon)$ for some $\varepsilon > 0$. Now let $\|x\| \le 1$ and $0 < \delta < \varepsilon$. Then $\|(\varepsilon-\delta)x\| < \varepsilon$ and hence

$$\|T(x)\| \;=\; (\varepsilon-\delta)^{-1}\|T(\varepsilon-\delta)(x)\| < (\varepsilon-\delta)^{-1} \;.$$

This shows that $\|T\|_{op} \le (\varepsilon-\delta)^{-1}$ for every $\delta > 0$ and thus $\|T\|_{op} \le \varepsilon^{-1}$. Now, we observe that $\| \; \|_{op}$ is a norm. We only check the triangle inequality. Indeed,

$$\begin{aligned}\|T+S\|_{op} &= \sup_{\|x\|\le 1} \|(T+S)(x)\| \;=\; \sup_{\|x\|\le 1} \|T(x)+S(x)\| \;\le\; \sup_{\|x\|\le 1} \|T(x)\| + \|S(x)\| \\ &\le \|T\|_{op} + \|S\|_{op} \;.\end{aligned}$$

Finally we have to show that $L(X,Y)$ is complete. Let (T_n) be a Cauchy sequence of linear maps. For fixed $x \in X$, we have

$$\|T_n(x) - T_m(x)\| \;\le\; \|T_n - T_m\|\|x\| \;.$$

Thus $(T_n(x))$ is Cauchy and we may define

$$T(x) \;=\; \lim_n T_n(x) \;.$$

Then we have

$$T(x+\lambda y) \;=\; \lim_n T_n(x+\lambda y) \;=\; \lim_n T_n(x) + \alpha T_n(y) \;=\; T(x) + \lambda T(y) \;.$$

Thus T is linear. Let us show that

op (4.1)
$$\lim_n \|T - T_n\|_{op} \;=\; 0 \;.$$

Indeed, let $x \in X$ with $\|x\| \le 1$. Then we have

$$\begin{aligned}\|T(x) - T_n(x)\| &= \|\lim_m T_m(x) - T_n(x)\| \;\le\; \limsup_{m\ge n} \|T_m(x) - T_n(x)\| \\ &\le \sup_{m\ge n} \|T_m - T_n\|\|x\| \;\le\; \sup_{m\ge n} \|T_m - T_n\| \;.\end{aligned}$$

In particular $\|T\|_{op} \leq \|T - T_1\|_{op} + \|T_1\|_{op}$ is finite and T is continuous. Moreover, $\lim_n d(T, T_n) = 0$ implies that $\lim_n T_n = T$. ■

Corollary 4.6. *Let X be a normed space. Then $X^* = L(X, \mathbb{R})$ is a Banach space. Moreover, $X^{**} = L(X, \mathbb{R})$ is a Banach space.*

Definition and Remark 4.7. *Let $\iota : X \to X^{**}$ be the linear map given by $\iota(x)(x^*) = x^*(x)$. Then*

$$\|\iota(x)\| \leq \|x\| .$$

Indeed, the Hahn-Banach theorem (proved in the next course) shows that $\|\iota(x)\| = \|x\|$. A Banach space X is called reflexive if $\iota(X) = X^{**}$, i.e. ι is surjective. All finite dimensional spaces are reflexive.

5. L_p spaces

In the following (Ω, Σ, μ) is a sigma-finite measure space. We define

$$\mathcal{L}_0 \;=\; \{f : \Omega \to \mathbb{R} \;:\; \lim_{\alpha\to\infty} \mu(|f| > \alpha) \;=\; 0\}\;.$$

On $\mathcal{L}_0$ we define the equivalence relation

$$f \sim g \quad \text{if } f \;=\; g \; \mu \; a.e.$$

i.e. there exists a set $F \in \Sigma$ with measure 0 such that $f(\omega) = g(\omega)$ for all $\omega \in F^c$. We define

$$L_0(\mu) \;=\; \mathcal{L}_0/\sim$$

Proposition 5.1. *(Hw) $L_0(\mu)$ equipped with the distance*

$$d([f],[g]) \;=\; \inf\{\varepsilon \;:\; \mu(|f-g| > \varepsilon) < \varepsilon\}$$

is a complete metric space.

Definition 5.2. *$\mathcal{L}_p$ is the set of all measurable functions f such that $\int |f|^p$ is finite.*

`mink` Lemma 5.3. *(Hölder's inequality) Let $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in \mathcal{L}_p$, $g \in \mathcal{L}_q$. Then fg is integrable and*

$$|\int fg| \;\le\; (\int |f|^p)^{\frac{1}{p}} (\int |q|^q)^{\frac{1}{q}}\;.$$

Proof. We use the fact that $g(x) = -\ln(x)$ is convex. Thus for positive numbers a, b we have

$$-\ln(\frac{a^p}{p} + \frac{b^q}{q}) \;=\; g(\frac{a^p}{p} + \frac{b^q}{q}) \;\le\; \frac{1}{p} g(a^p) + \frac{1}{q} g(b^q) \;=\; -\ln(a) - \ln(b)\;.$$

This yields

`minkk` (5.1)
$$ab \;\le\; \frac{a^p}{p} + \frac{b^q}{q}\;.$$

Thus for every $\omega \in \Omega$ and $s > 0$ we find

$$|f(\omega)||g(\omega)| \;\le\; \frac{|sf(\omega)|^p}{p} + \frac{|s^{-1}g(\omega)|^q}{q}\;.$$

Using this for $s = 1$, we deduce that $\int |f|^p < \infty$ and $\int |g|^q < \infty$ implies $\int |fg| < \infty$. Thus we get

$$|\int fg| \;\le\; \int |fg| \;\le\; \frac{s^p}{p} \int |f|^p + \frac{s^{-q}}{q} \int |g|^q\;.$$

We define $s = \frac{(\int |g|^q)^{1/(p+q)}}{(\int |f|^p)^{1/(p+q)}}$. This implies

$$s^p \int |f|^p \;=\; s^{-q} \int |g|^q \,.$$

Hence, we deduce from $p/(p+q) = 1/q$ and $q/(p+q) = 1/p$ that

$$\begin{aligned} |\int fg| \le s^p \int |f|^p \;&=\; (\int |g|^q)^{p/(p+q)} (\int |f|^p)^{1-p/(p+q)} \\ &= (\int |g|^q)^{1/q} (\int |f|^p)^{1/p} \,. \end{aligned}$$

■

`conv` Remark 5.4. *Let f be a measurable function. Then*

$$(\int |f|^p)^{1/p} \;=\; \sup\{ |\int fg| \;:\; g \in S(\mu) \int |g|^q \le 1 \} \,.$$

Proof. Let $0 \le h \le |f|^p$ be a simple function. This implies $0 \le h^{1/p} \le |f|$. We write $h^{1/p} \;=\; \sum_i r_i 1_{E_i}$ and define

$$g(\omega) \;=\; (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} 1_{E_i(\omega)} \frac{f(\omega)}{|f(\omega)|} \,.$$

Then, we get

$$\begin{aligned} \int fg \;&=\; (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} \int_{E_i} |f(\omega)| \\ &\ge\; (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} \int_{E_i} r_i \\ &= (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} r_i \mu(E_i) \\ &= (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^p \mu(E_i) \;=\; (\int h)^{\frac{1}{p}} \,. \end{aligned}$$

On the other hand

$$\int |g|^q d\mu = (\int |h|)^{-1} \sum_i r_i^p \mu(E_i) \;=\; 1 \,.$$

This yields the assertion. ■

`triang` Lemma 5.5. *Let f and g be measurable functions. Then*

$$(\int |f+g|^p)^{\frac{1}{p}} \;\le\; (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} \,.$$

PROOF. It suffices to consider $1 < p < \infty$. We may assume that the right hand is finite. Let $0 \le h \le |f+g|$. Then, we have

$$h^p \;=\; hh^{p-1} \;\le\; (|f|+|g|)h^{p-1} \,.$$

By Lemma `mink` 5.3 we deduce (for $\frac{1}{q} = 1 - \frac{1}{p}$)

$$\begin{aligned}
\int h^p &\le \int |f|h^{p-1} + \int |g|h^{p-1} \\
&\le (\int |f|^p)^{\frac{1}{p}} (\int h^{(p-1)q})^{1-\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} (\int h^{(p-1)q})^{1-\frac{1}{p}} \\
&= \Big((\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} \Big) (\int h^p)^{1-\frac{1}{p}} \,.
\end{aligned}$$

If $\int h^p = 0$ there is nothing to show. In the other case we obtain

$$(\int h^p)^{\frac{1}{p}} \;\le\; (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} \,.$$

Taking the sup over all $0 \le h \le |f+q|$ we deduce the assertion. ■

THEOREM 5.6. *Let $1 \le p < \infty$. The space*

$$L_p \;=\; \{[f] \,:\, f \textit{ measurable } , \|[f]\|_p \;=\; (\int |f|^p)^{\frac{1}{p}}\}$$

with the norm $\| \, \|_p$ is a Banach space.

PROOF. We note first that for $f \sim g$ we have

$$\int |f|^p \;=\; \int |g|^p \,.$$

Thus $\| \, \|_p$ is well-defined. Moreover, we have

$$\|[\lambda f]\|_p \;=\; |\lambda| \|[f]\|_p \,.$$

By the definition of equivalent classes, we see that

$$\|[f]\|_p \;=\; 0 \Leftrightarrow f = 0 \; a.e. \Leftrightarrow [f] = 0 \,.$$

For $x, y \in L_p$, we pick $f \in x$, $g \in y$. According to Lemma `triang` 5.5, we deduce that $|f+g|^p$ is integrable and hence $x + y = [f+g]$ is in L_p satisfying

$$\|x+y\|_p \;=\; \|[f+g]\|_p \;=\; (\int |f+g|^p)^{\frac{1}{p}} \;\le\; (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} \;=\; \|x\|_p + \|y\|_p \,.$$

Thus $(L_p, \| \, \|_p)$ is a normed vector space. Let (x_n) be a Cauchy sequence in L_p. We may assume

$$\|x_n - x_{n+1}\|_p \;\le\; 2^{-n\frac{p+1}{p}} \,.$$

Let $f_n \in x_n$. Then we find

$$\int |f_n - f_{n+1}|^p \;\le\; 2^{-n(p+1)} \,.$$

By Chebychev's inequality we deduce

$$\mu(|f_n - f_{n+1}| > 2^{-n})2^{-np} \;\le\; \int |f_n - f_{n+1}|^p \;\le\; 2^{-n(p+1)} \,.$$

Thus we get

$$\mu(|f_n - f_{n+1}| > 2^{-n}) \;\le\; 2^{-n} \,.$$

The convergence Lemma implies that (f_n) is almost everywhere convergent to a measurable function f. Define

$$h \;=\; |f_1| + \sum_n |f_{n+1} - f_n| \,.$$

We want to show that $|h|^p$ is integrable. Indeed, we apply the monotone convergence Lemma and deduce from the triangle inequality in $\mathcal{L}_p$

$$\begin{aligned} \int |h|^p &\le \liminf_m \int (|f_1| + \sum_{n=1}^m |f_{n+1} - f_n|)^p \\ &\le \liminf_m (\|[f_1]\|_p + \sum_{n=1}^m \|[f_{n+1} - f_n\|_p)^p \\ &\le (\|x_1\|_p + \sum_n \|x_{n+1} - x_n\|_p)^p \;\le\; (\|x_1\|_p + 2)^p \;< \infty \,. \end{aligned}$$

Moreover, we have $|f_n - f_m|^p \;\le\; |h|^p$ and for $m \ge n$ we get that

$$(\int |f_m - f_n|^p)^{\frac{1}{p}} \;\le\; \sum_{k=n}^m \|x_{k+1} - x_k\|_p \;\le\; \sum_{k=n}^\infty 2^{-k\frac{p+1}{p}} \;\le\; 2^{1-n} \,.$$

By the dominated convergence theorem (with majorant $|h|^p$) we deduce that

$$\int |f - f_n|^p d\mu \;=\; \lim_{m \ge n} \int |f_m - f_n|^p \;\le\; 2^{p(1-n)} \,.$$

Using the triangle and $f_1 \in \mathcal{L}_p$, we deduce that $f \in L_p$. Moreover,

$$\lim_n \|[f] - x_n\|_p \;=\; 0 \,.$$

This completes the proof. ∎

Proposition 5.7. *Let $1 \le p < \infty$. The simple functions are dense in L_p. For $(\Omega, \Sigma, \mu) = (\mathbb{R}, \mathcal{L}, m)$ the step functions are dense in L_p and the continuous functions are dense in L_p.*

Proof. Let $f \geq 0$ such that $\int f^p < \infty$. Let h_n be an increasing sequence of simple functions such that

$$I(f^p - h_n) < 4^{-n} \,.$$

Then h_n converges to f^p a.e. and also $h^{\frac{1}{p}}$ converges to f almost everywhere. Since $|f - h_n|^p \;\leq\; |f|^p$, we deduce from the dominated convergence theorem that

$$\lim_n \int |f - h_n|^p \;=\; \int 0 \;=\; 0 \,.$$

This proves $\lim_n \|[f] - [h_n]\|_p = 0$. The general assertion follows by considering $f = f^+ - f^-$. For the second assertion, we assume again that $f \geq 0$ and $0 \leq h \leq f$ such that

$$\int |f - h|^p < \varepsilon \,.$$

Let $C = \sup \|h\|$. Using a small perturbation, we may also assume that h vanishes in $(-\infty, n] \cup [n, \infty)$. By the applications of Lusin's theorem, we may find a step function g such that $0 \leq g \leq C$ and

$$mu(|g - h| > \delta) < \delta \,.$$

Then, we get

$$\begin{aligned} \int_{-n}^{n} |g - h|^p &= \int_{-n}^{n} 1_{|g-h|>\delta} |g - h|^p + \int_{-n}^{n} 1_{|g-h|\leq\delta} |g - h|^p \\ &\leq (2C)^p m(|g - h| > \delta) + \delta^p 2n \;\leq\; (2C)^p \delta + 2n\delta^p \,. \end{aligned}$$

Choosing δ small enough we get $(\int_{-n}^{n} |g - h|^p)^{\frac{1}{p}} < \varepsilon$. Starting from step functions, continuous functions are achieved as in Lemma 2.8. ■