1. Some remarks on the ternary Cantor function

The complement of the Cantor set in $[0,1]$ is given by

$$O = \bigcup_{n=0}^{\infty} \bigcup_{a\in\{0,2\}^n} O_a$$

where for $n = 0$, $a = \emptyset$

$$O_a = (\frac{1}{3}, \frac{2}{3})$$

and for $n \geq 0$, $a = (a_1, ..., a_n)$ we have

$$O_a = (\sum_{i=1}^{n} a_i 3^{-i} + 3^{-(n+1)}, \sum_{i=1}^{n} a_i 3^{-i} + 2 3^{-(n+1)}) .$$

On such a set O_a we define

$$f(x) = \sum_{i=1}^{n} \frac{a_i}{2} 2^{-i} + 2^{-(n+1)}$$

For $n = 0$ we use the value $\frac{1}{2}$.

Lemma 1.1. *f is uniformly continuous.*

Proof. Let us consider x_1 and x_2 in O such that $|x_1 - x_2| < 3^{-n}$. We may write

$$x_1 = \sum_{i=1}^{\infty} a_i 3^{-i}$$

and

$$x_2 = \sum_{i=1}^{\infty} b_i 3^{-i}$$

such that there exists a smallest integer k and m with $a_k = 1$, $b_m = 1$. Let j be the smallest integer such that $a_j \neq b_j$.

case 1) $j < \min(k,m)$: W.l.o.g. we may assume $a_j = 2$ and $b_j = 0$. Then

$$3^{-n} \geq x_1 - x_2 \geq 2 3^{-j} - 3^{-j} = 3^{-j} .$$

This yields $j \geq n$ and

$$|f(x_1) - f(x_2)| \leq \sum_{i\leq j} |a_i - b_i| 2^{-i} \leq 2 2^{-j} \leq 4 2^{-n} .$$

case 2) $j \geq \min(k,m)$. If $k = m$, then $f(x_1) = f(x_2)$ are we are fine. Let us assume $k > m$. Then $a_m \in \{0,2\}$ and $b_m = 1$. Thus $j = m$.

case 2a) $a_m = 2$: Let us fix the smallest $j > k$ such that $a_j \neq 0$. We get

$$f(x_1) - f(x_2) \;=\; \sum_{i=m+1}^{k-1} \frac{a_i}{2} 2^{-i} + 2^{-m} \,.$$

Moreover,

$$3^{-n} x_1 - x_2 \geq 3^{-k} + a_j 3^{-j} - (\sum_{i=k+1}^{\infty} b_i 3^{-i}) \geq a_j 3^{-j} \,.$$

Thus $j \geq n$ and hence

$$|f(x_1) - f(x_2)| \;\leq\; \sum_{i \geq j} \frac{a_i}{2} 2^{-i} + 2^{-m} \;\leq\; 4 2^{-j} \;\leq\; 4 2^{-n} \,.$$

case 2b) $a_m = 0$: similar. ■

COROLLARY 1.2. *There exists a continuous strictly increasing function* $g : [0,1] \to [0,2]$ *such that* $m(C) = 1$.

PROOF. Let F be the unique extension of f to $[0,1]$. One can show that f is increasing and $f(0) = 0$, $f(1) = 1$. Then $g(x) = F(x) + x$ is strictly increasing and by the intermediate value Theorem we have $g([0,1]) = [0,2]$. Moreover, $g(O_n)$ is an interval with length $3^{-(n+1)}$. Thus yields $m(g(O)) = 1$. Taking complements yields the assertion. ■

COROLLARY 1.3. *There exists a Lebesgue measurable set which is not borel.*

PROOF. Let $D \subset m(C)$ be a non-measurable set. Assume that $B = g^{-1}(D) \subset C$ is a borel set. Then we deduce that

$$D \;=\; g(B) \;=\; (g^{-1})^{-1}(B)$$

is also a borel set because g^{-1} is continuous (why). This contradiction shows that B is not a borel set. However, $B \subset C$ is a subset of a set of measure 0 and hence Lebesgue measurable. ■