2. A covering lemma and the local maximal function

Among the most important tools in analysis are the results known as covering theorems. For the real line, the covering sets are intervals. In this section, we present a simple form of the best covering theorem for the real line. The original result is an extension by J. Aldaz ["A general covering lemma for the real line", *Real Anal. Exchange* **17**(1991/92), 394–398] of a lemma of T. Radó ("Sur un problème relatif à un théorème de Vitali", *Fundamenta Mathematicae* **11**(1928), 228–229.) In the Aldaz result, the constant 3 is improved to $2+\varepsilon$ for an arbitrary $\varepsilon > 0$, and the result is valid for any finite Borel measure, not just Lebesgue measure m on a bounded interval.

THEOREM 2.1 (Rado-Aldaz). *Given an arbitrary collection $\mathcal{I}$ of non-degenerate intervals, all contained in a fixed bounded interval J, the set $\cup_{I\in\mathcal{I}} I$ is measurable, and there is a finite disjoint subset $\{I_1, \cdots, I_n\} \subseteq \mathcal{I}$ such that*

$$m(\cup_{I\in\mathcal{I}} I) \leq 3 \cdot \sum_{k=1}^{n} m(I_k).$$

PROOF. Let B be the set of those right-hand end points of intervals $I \in \mathcal{I}$ such that $b \in I$ but b is not in the *interior* of any interval of $\mathcal{I}$. Then for some $\delta > 0$, $(b-\delta, b) \subset I$ and $(b-\delta, b] \cap B = \{b\}$. We may associate a rational number in the interval $(b-\delta, b)$ with b. It follows that B is a countable set. A similar fact is true for left-hand end points that are not interior to any interval in $\mathcal{I}$. Therefore, $(\cup_{I\in\mathcal{I}} I) \setminus (\cup_{I\in\mathcal{I}} I^\circ)$ is at most a countable set. By Lindelöf's theorem (i.e., any union of open intervals equals the union of a countable subcollection) we may assume that $\mathcal{I}$ itself is a countable collection $\{I_n\}$, whence $\cup_{I\in\mathcal{I}} I = \cup_{n=1}^{\infty} I_n$ is measurable.
Now since all the intervals are contained in the bounded interval J,

$$m(\cup_{n=1}^{\infty} I_n) = \lim_{N\in\mathbb{N}} m(\cup_{n=1}^{N} I_n) < +\infty.$$

We employ Rado's result after first choosing N so that

$$\frac{3}{2} \cdot m(\cup_{n=1}^{N} I_n) \geq m(\cup_{n=1}^{\infty} I_n) = m(\cup_{I\in\mathcal{I}} I).$$

Ordering the finite collection, we discard the first interval if it is covered by the remaining intervals. Otherwise, we keep the first interval and consider the second. In either case, this does not change the measure $m(\cup_{n=1}^{N} I)$. Continuing in this way, we may assume that each I_n in our finite collection contains a point x not in any other interval of the collection. Now, we order these points and reorder the corresponding intervals so that for any indices i, j, and k with $i < j < k$ we have $x_i < x_j < x_k$ and thus $I_i \subseteq (-\infty, x_j)$ and $I_k \subseteq (x_j, +\infty)$. That is, the points are given the ordering inherited from $\mathbb{R}$, and the intervals are given the same ordering as their points. Since the intervals with even indices form a disjoint collection, as do the intervals with odd indices, the desired subset of $\mathcal{I}$ is whichever of these two families has the greater total measure. For example, if we choose the even indices, then

$$3 \cdot m(\cup I_{2n}) = \frac{3}{2} \cdot 2 \cdot m(\cup I_{2n}) \geq \frac{3}{2} \cdot m(\cup_{n=1}^{N} I_n) \geq m(\cup_{I \in \mathcal{I}} I). \qquad \blacksquare$$

The next result is refined maximal inequality due to Jürgen Bliedtner and P. Loeb ("Limit Theorems via Local Maximal Functions", preprint.) Here, we let f be a nonnegative integrable function on $\mathbb{R}$. We set $\mathcal{I}(x,r)$ equal to the set of intervals I containing x with positive length $m(I) \leq r$, and we set

$$M(f,r,x) := \sup_{I \in \mathcal{I}(x,r)} \frac{1}{m(I)} \int_I f \, dm.$$

Since $M(f,r,x)$ decreases as r decreases, we may set

$$M(f,x) := \lim_{r \to 0+} M(f,r,x),$$

where the limit is understood to be $+\infty$ if $M(f,r,x) = +\infty$ for all $r > 0$.

PROPOSITION 2.2. *Let E be a bounded subset of $\mathbb{R}$. Fix $\alpha > 0$, and let $E_\alpha = \{x \in E : M(f,x) > \alpha\}$. Then the outer measure of E_α satisfies*

$$m^*(E_\alpha) \leq \frac{3}{\alpha} \cdot \int_{\mathbb{R}} f \, dm.$$

PROOF. Given $x \in E_\alpha$, there is an interval $I_x \in \mathcal{I}(x,1)$ such that

$$\alpha \cdot m(I_x) \leq \int_{I_x} f \, dm.$$

These intervals form a collection $\mathcal{I}$ that cover E_α, so by Theorem 2.1, there is a finite disjoint subcollection $\{I_1, \cdots, I_n\} \subset \mathcal{I}$ such that

$$m^*(E_\alpha) \leq m(\cup_{I \in \mathcal{I}} I) \leq 3 \cdot \sum_{k=1}^{n} m(I_k) \leq \frac{3}{\alpha} \sum_{k=1}^{n} \int_{I_k} f \, dm \leq \frac{3}{\alpha} \cdot \int_{\mathbb{R}} f \, dm. \qquad \blacksquare$$

PROPOSITION 2.3. *Let E be a bounded measurable subset of $\mathbb{R}$, and let f be a nonnegative integrable function that vanishes almost everywhere on E. Then $M(f,x) = 0$ for almost all $x \in E$.*

PROOF. Given $\alpha > 0$ and an $\varepsilon > 0$, we may fix an open set $U \supseteq E$ so that $\int_U f\, dm < \varepsilon\alpha/3$. Now

$$E_\alpha := \{x \in E : M(f, x) > \alpha\} = \{x \in E : M(f \cdot \chi_U, x) > \alpha\}.$$

Therefore, $m^*(E_\alpha) \leq \frac{3}{\alpha} \int_U f \; dm < \varepsilon$. Since ε is arbitrary, $m^*(E_\alpha) = 0$, and the result follows. ∎