4. Hilbert spaces

A (real) Hilbert space comes with a vector space H, a scalar product $(\ ,\ ) : H \times H \to \mathbb{R}$ satisfying

$$\begin{aligned} \text{i)}\ & (x,x) \geq 0 \text{ and } (x,x) = 0 \text{ iff } x = 0, \\ \text{ii)}\ & (x,y) \ = \ (y,x) \text{ and} \\ \text{iii)}\ & (x, y+\lambda z) \ = \ (x,y) + \lambda(x,z), \end{aligned}$$

for all $x, y, z \in H$ and $\lambda \in \mathbb{R}$. Moreover, we require that H is complete with respect to the metric

$$d(x,y) \ = \ (x-y, x-y)^{\frac{1}{2}} \ .$$

(With completeness this is called a pre-Hilbert space.)

LEMMA 4.1. *$\|x\| = (x,x)^{\frac{1}{2}}$ is a norm. Moreover,*

$$\boxed{\texttt{CS}} \qquad |(x,y)| \ \leq \ \|x\|\|y\| \ . \tag{4.1}$$

PROOF. We first show

$$|(x,y)| \ \leq \ \|x\|\|y\| \ .$$

For this let $\lambda \in \mathbb{R}$. Then we have

$$0 \ \leq \ (x+\lambda y, x+\lambda y) \ = \ (x,x) + 2\lambda(x,y) + \lambda^2(y,y) \ .$$

This yields

$$2(-\lambda)(x,y) \ \leq \ (x,x) + \lambda^2(y,y) \ .$$

If $(x,y) \geq 0$ we define $\lambda = -\frac{\|x\|+\varepsilon}{\|y\|+\varepsilon}$. This yields

$$\begin{aligned} 2|(x,y)| &\leq \frac{\|y\|+\varepsilon}{\|x\|+\varepsilon}(\|x\|^2 + (\|x\|+\varepsilon)^2 \frac{\|y\|^2}{(\|y\|+\varepsilon)^2} \ \leq \ \frac{\|y\|+\varepsilon}{\|x\|+\varepsilon} 2(\|x\|+\varepsilon)^2 \\ &\leq 2(\|y\|+\varepsilon)(\|x\|+\varepsilon) \ . \end{aligned}$$

Letting $\varepsilon \to 0$ yields ($\overset{\texttt{CS}}{4.1}$). Now, we consider again $x, y \in H$. Then

$$\begin{aligned} \|x+y\|^2 = (x+y, x+y) \ &= \ (x,x) + 2(x,y) + (y,y) \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \ = \ (\|x\| + \|y\|)^2 \ . \end{aligned}$$

■

DEFINITION 4.2. *A system (x_i) is called an orthonormal if*

$$(x_i, x_j) \ = \ \delta_{ij} \ .$$

A system $(x_i) \subset H$ *is called* *orthonormal basis* *if moreover,*

$$\overline{\{\sum_i \lambda_i x_i \ : \ \lambda_i \in \mathbb{R}\}} \ = \ H \ .$$

Here we are taking only finite linear combinations.

`bessel` LEMMA 4.3. *Let* (x_i) *be an orthonormal system and* $x \in H$*. Then*

$$(\sum_i |(x_i,x)|^2)^{\frac{1}{2}} \ \le \ \|x\| \ .$$

Moreover, $y = \sum_i (x_i,x)x_i$ *is in the closure of* $\{\sum_i \lambda_i x_i\}$*.*

PROOF. Let $J \subset I$ be a finite subset. Let $\lambda_i \ = \ (x_i,x)$ and consider

$$y \ = \ \sum_{i\in J} \lambda_i x_i \ .$$

Then, we have

$$(y,y) \ = \ \sum_{i,j} \lambda_i \lambda_j (x_j,x_i) \ = \ \sum_j |\lambda_j|^2 \ .$$

By ($\overset{\texttt{CS}}{4.1}$) we deduce

$$\sum_{j\in J} |(x_i,x)|^2 = |(x,y)| \ \le \ \|x\| (\sum_i \|(x_i,x)|^2)^{\frac{1}{2}} \ .$$

Cancellation yields

$$(\sum_{i\in J} |(x_i,x)|^2)^{\frac{1}{2}} \|x\| \ .$$

Note that

$$\sum_i |(x_i,x)|^2 \ = \ \sup_{J\subset I finite} \sum_{i\in J} |(x_i,x)|^2$$

is the considered as a definition here. For the second assertion, we consider $I_n = \{i \in I \ : \ |(x_i,x)| > \frac{1}{n}\}$. Note that I_n has to be finite set. Thus $I' = \bigcup_n I_n$ is countable set. We may assume $I' = \mathbb{N}$ and

$$\sum_n |(x_{i_n},x)|^2 < \infty \ .$$

Thus for every $\varepsilon > 0$ we may find n_0 such that for every $m > n > n_0$ we have

$$\sum_{k=n}^{m} |(x_{i_k},x)|^2 < \varepsilon \ .$$

By the above, we deduce that

$$y_n \ = \ \sum_{k=1}^{n} (x_{i_k},x) x_{i_k}$$

is Cauchy and that the limit y satisfies $(x_i, y) = (x_i, x)$ and

$$\lim_n \|y - y_n\| \;=\; 0 \,.$$

This complete the proof. ■

Remark 4.4. *The element* $y = \sum_i (x_i, x) x_i$ *satisfies*

$$\inf_{z \in cl(\{\sum_i \lambda_i x_i\})} \|x - z\| \;=\; \|x - y\| \,.$$

Lemma 4.5. *Let* (x_i) *be orthonormal. Then there exists an orthonormal basis containing* (x_i)*.*

Proof. Let S be the collection of orthonormal sets containing (x_i). It is easily checked that for every chain the union is an element in S. By Zorn's Lemma we may find a maximal element (y_j) in S. Let us assume that $x \in H$ does no belong to the closure of $\{\sum_j \lambda_j y_j\}$. Then, we define

$$z \;=\; x - \sum_j (y_j, x) y_j$$

and $y = z/\|z\|$. It is easily seen that y has norm 1 and $(y, y_j) = 0$ for all j. Thus we may add y to the system (y_j). This contradiction concludes the proof. ■

Corollary 4.6. *The dual of* H *is* H *and* H *is reflexive.*

Proof. Let $f : H \to \mathbb{C}$ be a linear functional. Let (x_i) be an ONS. We define

$$\lambda_i \;=\; f(x_i) \,.$$

Let $J \in I$ be a finite subset. Then

$$\begin{aligned} \sum_{i \in J} |\lambda_i|^2 = |f(\sum_{i \in J} \lambda_i x_i)| \;&\le\; \|f\| \|\sum_i \lambda_i x_i\| \\ &\le \|f\| (\sum_i \|\lambda_i\|^2)^{\frac{1}{2}} \,. \end{aligned}$$

Thus we deduce

$$(\sum_{i \in I} |\lambda_i|^2)^{\frac{1}{2}} \;\le\; \|f\| \,.$$

We have seen in Lemma `bessel` 4.3 that $y = \sum_i \lambda_i x_i$ converges and hence

$$(y, x_i) \;=\; \lambda_i \,.$$

Thus f and the functional $f_y(x) = (y, x)$ coincide on a dense set and are Lipschitz. By the unique extension principle they coincide. Thus the dual H^* of H is exactly given by functionals $f_y(x) = (y,x)$ and $\|f_y\| = \|y\|$. Hence the dual of H^* is given by functionals $g_z(f_y) \;=\; (z,y)$, $z \in H$. This implies $H^{**} = H$. ■

COROLLARY 4.7. *The dual of* $L_2(\Omega,\Sigma,\mu)$ *is* $L_2(\Omega,\Sigma,\mu)$.

PROOF. We note that

$$(f,g) \;=\; \int fg$$

is a scalar product and $\|f\|_2 \;=\; (f,f)^{\frac{1}{2}}$. Since $L_2(\Omega,\Sigma,\mu)$ is complete we deduce that $L_2(\Omega,\Sigma,\mu)$ is Hilbert space. Thus the dual is given by the linear functionals $\phi_f : L_2(\Omega,\Sigma,\mu) \to \mathbb{R}$ defined as

$$\phi_f(g) \;=\; \int fg \,.$$

This yields the assertion. ■

For the next duality result we will have to introduce the space $L_\infty(\Omega,\Sigma,\mu)$ of equivalence class with the norm

$$\|f\|_\infty \;=\; \inf_{\mu(E)=0} \sup_{\omega\in E^c} |f(\omega)| \,.$$

LEMMA 4.8. *Let* f *be a measurable function. Then*

$$\|f\|_\infty \;=\; \sup_{\int |g|\le 1} |\int fg| \,.$$

PROOF. Let $\|f\|_\infty \;<\; 1$. Then there exists a set E of measure 0 such that $|f(\omega)| \le 1$ for all $\omega in E^c$. Thus we get by monotonicity

$$|\int fg| \;\le\; \int_{E^c} |f||g| \;\le\; \int_{E^c} |g| \;\le\; \int |g| d\mu \,.$$

For the converse we assume $\|f\|_\infty \;>\; r$. Let $E \;=\; \{\omega \;:\; |f| \;>\; r\}$. Then we must $\mu(E) \;>\; 0$. Since Ω is σ-finite we find a subset $F \subset E$ with $0 \;<\; \mu(F) \;<\; \infty$. We define

$$g(\omega) \;=\; \mu(F)^{-1} 1_F \frac{f(\omega)}{|f(\omega)|} \,.$$

Then $\int |g| \;=\; 1$ and

$$\int fg \;=\; \mu(F)^{-1} \int_F |f| \;\ge\; r \,.$$

This yields the assertion. ■

`ddual` COROLLARY 4.9. *Let* $1 \le p \le 2$ *and* (Ω,Σ,μ) *be probability space. Then the dual space of* $L_p(\Omega,\Sigma,\mu)$ *is* $L_{p'}(\Omega,\Sigma,\mu)$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let us first consider a function $f \in L_2$. We define $r = 2/p \geq 1$. Then Hölder's inequality implies

$$\int |f|^p d\mu = (\int |f|^{pr})^{\frac{1}{r}} (\int 1 d\mu)^{1-\frac{1}{r}} \; = \; (\int |f|^2)^{\frac{1}{r}} \, .$$

This implies

$$\|f\|_p \; \leq \; \|f\|_2 \, .$$

Now, let $\phi : L_p \to \mathbb{R}$ be a continuous linear functional. Then ϕ is a continuous linear functional on L_2. Thus we can find a function $g \in L_2$ such that

$$\phi(f) \; = \; \int g f d\mu$$

holds for all $g \in L_2$. Let h be a measurable function such that $|h|$ is a simple function. Then $h \in L_2$ and hence Remark 1.9. implies

$$(\int \||g|^{p'} d\mu)^{\frac{1}{p'}} \; = \sup_{|h| \in S(\mu) \|h\|_p \leq 1} | \int g h d\mu| \; \leq \; \|\phi\| \, .$$

Thus $[g] \in L_{p'}(\Omega, \sigma, \mu)$. By Hölder's inequality we deduce

$$\phi_g(f) \; = \; \int g f d\mu$$

is a continuous linear functional. Moreover, for every simple function h we deduce from $h \in L_2$ that

$$\phi(h) \; = \; \phi_g(h) \, .$$

By density of the simple functions (and the unique extension principle) ϕ and ϕ_g coincide. ■

Remark 4.10. *The previous result extends easily to σ-finite measure spaces and is true in full generality.*

5. Radon-Nikodym Theorem

Definition 5.1. *Let Σ be a σ-algebra and ν and μ measures on Σ. We say that ν is absolutely continuous with respect to μ (in short $\nu \ll \mu$) if $\mu(E) = 0$ implies $\nu(E) = 0$.*

Theorem 5.2. *Let μ and ν be σ-finite measure such that $\nu \ll \mu$. Then there exists a positive measurable function g such that*

$$\nu(E) \;=\; \int_E g d\mu \,.$$

Proof. We consider the measure $\mu_1 \;=\; \mu + \nu$. It is easily check that μ_1 is also σ-finite. Therefore we assume $\mu_1(\Omega) < \infty$. Then the linear functional $\phi : L_1(\mu_1) \to \mathbb{R}$ defined by

$$\phi(f) \;=\; \int f d\nu \,.$$

For a positive simple function $f = \sum_i a_i 1_{E_i}$ we have

$$\phi(f) \;=\; \sum_i a_i \nu(E_i) \;\le\; \sum_i a_i \mu_1(E_i) \;=\; \|f\|_1 \,.$$

By density ϕ extends to a function of norm ≤ 1. According to Corollary ddual 4.9 we find $g \in L_\infty(\mu_1)$ such that

$$\phi(f) \;=\; \int g f d\mu \,.$$

It is easily checked that g is positive and hence $0 \le g \le 1$. Thus, we deduce inductively

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$$\begin{aligned} \int f d\nu &= \int f g d\mu + \int f g d\nu \\ &= \int f g d\mu + \int f g^2 d\mu + \int f g^2 d\nu \\ &= \int f g d\mu + \int f g^2 d\mu + \int f g^3 d\mu + \int f g^3 d\nu \\ &= \int f (\sum_{k=1}^n g^k) d\mu + \int f g^n d\nu \,. \end{aligned} \tag{5.1}$$

Let us consider the set $E_m = \{\omega : 0 \le g(\omega) < 1 - \frac{1}{m}\}$. Passing to the limit we deduce that

$$\int_{E_m} f d\mu \;=\; \lim_n \int_{E_m} f(\sum_{k=1}^n g^k) d\mu + \int f g^n d\nu \;=\; \int_{E_m} f g (1-g)^{-1} d\mu \,.$$

Let $E = \bigcup_m E_m$. By the dominated convergence theorem we deduce that for every $F \subset \Omega$ we have

$$\int_{F\cap E} d\nu \;=\; \lim_m \int_{F\cap E_m} d\nu \;=\; \lim_m \int_{F\cap E_m} (1-g)^{-1} d\mu \;=\; \int_{E\cap F} g(1-g)^{-1} d\mu \,.$$

Let us consider the set E^c where $g = 1$. Then we deduce from (5.1) that

$$\int_E d\nu \;=\; \int_E d\mu + \int\limits_E d\nu$$

Thus $\mu(E) = 0$. By absolute continuity we get $\nu(E) = 0$. Therefore

$$\int_F d\nu \;=\; \int_F 1_E g(1-g)^{-1} d\mu$$

holds for every $F \in \Sigma$. ■

COROLLARY 5.3. *Let* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{p'}$. *The dual space of* L_p *is* $L_{p'}$.

PROOF. Let us assume μ finite. Let $\phi : L_p(\mu) \to \mathbb{R}$ be a continuous linear functional. We define

$$\nu(E) \;=\; \sup\{\sum_j |\phi(1_{E_j})| \;:\; E \;=\; \dot{\bigcup_j} E_j \;,\; E_j's \text{ disjoint } \} \,.$$

Clearly, $E_1 \subset E_2$ implies $\nu(E_1) \le \nu(E_2)$. Moreover, let $E = \dot{\bigcup}_j E_j$ and $\varepsilon_j = \frac{\phi(1_{E_j})}{|\phi(1_{E_j})|}$. Note that

$$h_n \;=\; \sum_{j\le n} \varepsilon_j 1_{E_j}$$

satisfies $|h_n| \le 1_\Omega$ and hence the dominated convergence theorem implies

$$\lim_n \| \sum_{j\ge n} \varepsilon_j 1_{E_j} \|_p \;=\; 0 \,.$$

Since ϕ is continuous we deduce

$$\sum_j |\phi(E_j)| \;=\; |\phi(\sum_j \varepsilon_j 1_{E_j})| \;\le\; \|\phi\| \| \sum_j \varepsilon_j 1_{E_j} \|_p \;\le\; \|\phi\| \mu(E)^{\frac{1}{p}} \,.$$

Thus $\nu(E) \le \mu(E)^{\frac{1}{p}}$. Let us show that ν is σ-additive. Indeed, let (F_i) be disjoint and $F_i \;=\; \cdot \bigcup E_{ij}$. Then

$$\nu(\bigcup_j F_j) \;\ge\; \sum_{i,j} |\phi(E_{ij})| \,.$$

Taking the supremum we get

$$\nu(\bigcup_j F_j) \;\ge\; \sum_i \nu(E_i) \,.$$

Conversely $\bigcup_j F_j = \bigcup_i E_i$. Then $F_j = \dot{\bigcup}_i F_j \cap E_i$. And hence

$$\sum_i |\phi(1_{E_i})| \leq \sum_{i,j} |\phi(1_{E_i \cap F_j})| \leq \sum_j \nu(F_j) \, .$$

Thus ν is a σ-additive measure which is absolutely continuous with respect to μ. This we find a positive function g such that

$$\nu(E) = \int g d\mu \, .$$

Now, we consider $\Phi : L_1(\nu) \to \mathbb{R}$ defined by

$$\Phi(\sum_i a_i 1_{E_i}) = \sum_i a_i \phi(1_{E_i}) \, .$$

Then

$$|\Phi(\sum_i a_i 1_{E_i})| \leq \sum_i |a_i| \nu(E_i)$$

Thus ϕ is a continuous functional and hence we find a measurable function h such that

$$\phi(\sum_i a_i 1_{E_i}) = \sum_i \int_{E_i} a_i h d\nu = \sum_i \int_{E_i} a_i h g d\mu = \int (\sum_i a_i 1_{E_i}) h g d\mu \, .$$

Since the simple functions are dense, we deduce from Remark 1.9. that $\|hg\|_{p'} \leq \|\phi\|$ and

$$\phi(f) = \int f h g d\mu \, .$$

This assertion is proved. ■

Corollary 5.4. *Let $1 < p < \infty$. Then L_p is reflexive.*

Proof. Indeed, $L_p^{**} = L_{p'}^{*} = L_p$. ■