Introduction to real analysis -hw1

Due date: Wednesday, September 8

i) [30P] Let $\mathbb{Q}$ denote the rational numbers. We say that a sequence (x_n) is Cauchy if

$$\forall_{k\in\mathbb{N}}\exists_{n_0\in\mathbb{N}}\forall_{n>m>n_0} \ : \ |x_n - x_m| < \frac{1}{k} \, .$$

Similarly, we say that (x_n) converges to 0 if

$$\forall_{k\in\mathbb{N}}\exists_{n_0\in\mathbb{N}}\forall_{n>n_0} \ : \ |x_n| < \frac{1}{k} \, .$$

We say that $(x_n) \sim (y_n)$ if $(x_n - y_n)$ converges to 0.

(a) Show that $\sim$ is an equivalence relation on the set $\mathbb{Q}^{\mathbb{N}}$ of all rational sequences.

(b) Let $X \subset \mathbb{Q}^{\mathbb{N}}$ the set of all Cauchy sequences in $\mathbb{Q}$. We denote by

$$[(x_n)] \ = \ \{(y_n) \in X : (y_n) \sim (x_n)\}$$

the equivalence associated to (x_n). Show that

$$[(x_n)] \cdot [(y_n)] \ = \ [(x_n y_n)]$$

is well-defined (this means $(x_n) \sim (z_n)$ and $(y_n) \sim (v_n)$ implies $(x_n yn) \sim (z_n v_n)$.

(c) Show that if $(x_n) \in X$ and $(x_n) \neq\sim (0)$ then there exists a Cauchy sequence (y_n) such that $(x_n y_n) \sim (1)$.

(d) Show that X is complete. More precisely, if $([(x_n^k)])_k$ is a Cauchy sequence (this means

$$\forall_{m\in\mathbb{N}}\exists_{k_0\in\mathbb{N}}\forall_{k_1>k_2>k_0}\exists_{n_0}\forall_{n>n_0}|x_n^{k_1} - x_n^{k_2}| < \frac{1}{m}) \, ,$$

then there exists a sequence $(x_n) \in X$ such that

$$\forall_{m\in\mathbb{N}}\exists_{k_0\in\mathbb{N}}\forall_{k>k_0}\exists_{n_0}\forall_{n>n_0}|x_n^k - x_n| < \frac{1}{m} \, .$$

(Hint: If you assume that

$$\forall_{k\in\mathbb{N}}\exists_{n_0}\forall_{n>n_0}|x_n^k - x_n^{k+1}| < 2^{-k}$$

than you find a suitable increasing subsequence (n_k) such that for $n_k < n \leq n_{k+1}$ you may define $x_n = x_n^k$.)

ii) [10p] Let $(F, +, \cdot, 0, 1, <)$ be an totally ordered field such that every Cauchy sequence converges. Show that every subset with an upper bound has a supremum.

iii) [10p] Show that if (x_n) is Cauchy, then (x_n) is bounded and

$$\limsup_n x_n = \lim_n x_n .$$

iv) [20p] Let $X = \mathbb{R}^2$ and

$$d_{SNCF}(x,y) = \begin{cases} d_2(x,y) & \text{if there exists } z \text{ and } t,s \in \mathbb{R} \\ & \text{such that } x = sz \text{ and } y = tz \\ d_2(x,(0,0)) + d_2((0,0),y) & \text{else} \end{cases} .$$

Show that $(\mathbb{R}, d_{SNCF})$ is a complete metric space. (Hint $d_2 \leq d_{SNCF}$ and the function $f(y) = d_{SNCF}(x,y)$ is lower semicontinuous.