

Real Analysis-Homework 10

Due date: Wednesday, December 1

- (1) (a) Let $x_1, \dots, x_n \in \mathbb{R}$. Show that

$$\sup_i |x_i| \leq \left(\sum_{i=1, \dots, n} |x_i|^p \right)^{\frac{1}{p}} \leq n^{\frac{1}{p}} \sup_{i=1}^n |x_i|.$$

- (b) Let $f = \sum_i x_i 1_{E_i}$ be a simple function. Show that

$$\lim_p \|f\|_p = \|f\|_\infty.$$

- (c) Let f be measurable function such that $[f] \in L_\infty$ show that

$$\lim_p \|f\|_p = \|f\|_\infty.$$

Hint: For a finite measure space you may use suitable simple function to approximate f from below and above. The general σ -finite case requires an additional approximation argument.

- (2) (a) Let $V = \{(x_n) : \exists k \in \mathbb{N} \forall n > k x_n = 0\}$ the space of finite sequences. We use

$$\|(x_n)\|_\infty = \sup_n |x_n|.$$

on V . Show that

$$\phi((x_n)) = \sum_n x_n$$

is a linear map on V which is not continuous.

- (b) Let $(V, \|\cdot\|)$ be a normed space and $\phi : V \rightarrow \mathbb{R}$ be a linear map. Show that ϕ is continuous if and only if

$$\{x \in V : \phi(x) = 0\}$$

is closed. (Hint: One implication is easy. For the other implication you may assume that there is a $x_0 \in V$ with $\phi(x_0) = 1$. Then $d = \inf\{\|x_0 - y\| : \phi(y) = 0\} > 0$ (why?). Use this to show that for arbitrary x with $\phi(x) = 1$ we have $\|x\| \geq d$ because $\phi(x - x_0) = 0$. Conclude).