

Introduction to real analysis -hw2

**Due date:** Monday, September 13

i) (30p) Show that

$$B = \{x \in \mathbb{R}^2 : d_{SNCF}(x, (0, 0)) \leq 1\}$$

is closed bounded, but not compact (in  $(\mathbb{R}^2, d_{SNCF})$ ). Let  $u : B \rightarrow (\mathbb{R}^m, d_2)$  be an injective map such that

$$d_2(u(x), u(y)) \leq d_{SNCF}(x, y).$$

Show that are sequences  $(x_n), (y_n) \subset B$  such that

$$\lim_n \frac{d_2(u(x_n), u(y_n))}{d_{SNCF}(x_n, y_n)} = 0.$$

(Hint: If this is not the case  $\inf \frac{d_2(u(x), u(y))}{d_{SNCF}(x, y)} \geq c > 0$  (why?). However, this leads to a contradiction if you have done your job before correctly).

ii) (20P) Let  $(r_n) \subset [0, 1]$  be a strictly increasing sequence. Let  $(\alpha_n) \subset [0, 1]$  be such that  $\sum_n \alpha_n$ . We define

$$f_n(x) = \alpha_n 1_{[r_n, 1]}(x) = \alpha_n \begin{cases} 1 & \text{if } x \geq r_n \\ 0 & \text{else} \end{cases}.$$

Show that for every  $x \in [0, 1]$

$$f(x) = \sum_n f_n(x)$$

is well-defined. Show that  $f$  is upper semi-continuous. Let  $y \in \mathbb{R}^2$ . Show that  $f(x) = -d_{SNCF}(x, y)$  is upper semicontinuous.

iii) (30p) Show that  $C[0, 1]$  with is not complete with respect to

$$d_1(f, g) = \int_0^1 |f(t) - g(t)|.$$

iv) (10p) Show that a closed subset of a compact set is compact. (If possible use the definition with open covers).

v) (10P). Show Lindelöf's theorem in  $\mathbb{R}^m$ : Every open subset of  $\mathbb{R}^m$  is a countable union of open balls.