Introduction to real analysis -hw3

Due date: Wednesday, September 20

(1) (20P) Show that $[0,1]$ is compact by verifying the definition.

(2) i) (10P) Given an example of a continuous function and a closed set C such that $f(C)$ is not closed.

ii) (10P) Show that image of a compact set under a continuous map is compact.

ii) (15P) Show that image of a relatively compact set under a continuous map is relatively compact.

iv) (10P) Let X be a compact metric space and $f : X \to Y$ be continuous and bijective. Show that the inverse function f^{-1} is continuous.

(3) (30P) Let us consider the set

$$Lip_c \ = \ \{f \in C[0,1] \ : \ |f(x) - f(y)| \leq c|x-y|\} \ .$$

Show that Lip_c is not relatively compact but

$$F \ = \ Lip_c \cap B(0,1) \ = \ \{f \in Lip_c \ : \ \sup_{0 \leq x \leq 1} |f(x)| \leq 1\}$$

is relatively compact in $C[0,1]$. As an application, we consider $I : C[0,1] \to C[0,1]$ given by

$$I(f)(t) \ = \ \int_0^t f(s) ds \ .$$

Show that

$$\{I(f) \ : \ \sup_{0 \leq x \leq 1} |f(x)| \leq 1\}$$

is relatively compact in $C[0,1]$.

(4) (20P) No 36a) on page=161: A point x in a metric space is called isolated if the set $\{x\}$ is open. Show that a complete metric space without isolated points has an uncountable number of points.