Introduction to real analysis -hw5

Due date: Monday, October 4

(1) (15P) Show that for set $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and $\varepsilon > 0$ there exists a compact set $C \subset E$ such that $m(C) \leq m(E) < m(C) + \varepsilon$. Conclude that for measurable E we have $m(E \setminus C) < \varepsilon$.

(2) (20P) Let μ be a σ-additive measure on $\mathcal{B}$ and (E_j) events in B,

  (a) Show that if $E_1 \supset E_2 \cdots$ and $m(E_1) < \infty$, then

$$\mu(\bigcap_j E_j) \; = \; \lim_j \mu(E_j) \, .$$

  Show that the assumption $\mu(E_1) < \infty$ is really needed.

  (b) Show that if $E_1 \subset E_2 \cdots$, then

$$\mu(\bigcup_j E_j) \; = \; \lim_j \mu(E_j) \, .$$

(3) (30P) Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone increasing function such that $f(x) = \lim_{y<x, y\to x} f(y)$. (That means f is continuous from the left.) We define

$$m_f([a,b)) \; = \; f(b) - f(a) \, .$$

Show that for every interval $I = [a,b]$ we have

$$f(b) - f(a) \; \leq \; \inf\{\sum_j m_f(I_j) \; : \; I \subset \bigcup_j I_j\} \, .$$

Here the infimum is taken over right open intervals $I_j = [a_j, b_j)$ (in principle we allow $a_j = -\infty$ and consider $(-\infty, b_j)$ as half open). Show that

$$f(b) - f(a) \; = \; \inf\{\sum_j m_f(I_j) \; : \; [a,b) \subset \bigcup_j I_j\} \, .$$

(4) (30P) We will need an estimate using Sterling's formula, namely

$$\lim_n 2^{-2n} \binom{2n}{n} \; = \; 0 \, .$$

(You can amuse yourself in finding Sterling's formula and how to deduce that.)

  (a) Let $X_n = \{-1,1\}^n$ and $\mu_n(A) = 2^{-n}|A|$ (where $|A|$ is the cardinality of A). Show that

$$\mu_n(\{(\varepsilon_1, ..., \varepsilon_n) : \sum_{i=1}^n \varepsilon_i = k\}) \; = \; 2^{-n} \binom{n}{\frac{n+k}{2}} \, .$$

Deduce from this that for even n

$$\mu_n(\{(\varepsilon_1, ..., \varepsilon_n) : |\sum_{i=1}^{n} \varepsilon_i| \leq k\}) \;=\; 2k2^{-n}\binom{n}{\frac{n}{2}}\,.$$

(b) On $X_\infty = \{-1, 1\}^\infty$ we denote by μ the extension of then μ_n's (explained in class). Show that for every k

$$\mu(\{(\varepsilon_1, ...) \;:\; \sup_n |\sum_{j=1}^{n} \varepsilon_j| \;\leq\; k\}) \;=\; 0\,.$$

(c) Let X_∞. Show that

$$\mu(\{(\varepsilon_1, ...) \;:\; \lim_n \sum_{j=1}^{n} \varepsilon_j \text{ exists }\}) \;=\; 0\,.$$