

Real Analysis-Homework 8

Due date: Monday, November 1

- (1) (15P) Let $\Omega = \mathbb{R}$, $\Sigma = \{A : A \text{ countable or } A^c \text{ countable}\}$ and

$$\mu(A) = \begin{cases} \infty & A^c \text{ is countable} \\ 0 & A \text{ is countable} \end{cases}.$$

Show that μ is σ -additive. Consider $f = 1_{\mathbb{R}}$. Show that

$$I(f) = 0 \quad \mu(\{x \in \mathbb{R} : f(x) \geq \frac{1}{2}\}) = \infty.$$

- (2) Let (Ω, Σ, μ) be a measure space (not necessarily σ -finite). We will now say that a function $f : \Omega \rightarrow [-\infty, \infty]$ is measurable in the strong sense if there exists a set $F \in \Sigma$ of measure 0 and a sequence (g_n) of simple functions such that

$$f(\omega) = \lim_n g_n(\omega)$$

holds for all $\omega \in F^c$.

- (a) (15P) Let $f \geq 0$ and measurable in the strong sense. Show that for every $\lambda > 0$ the set $E_\lambda = \{\omega \in \Omega : f(\omega) \geq \lambda\}$ is σ -finite, i.e. there exists $G_n \in \Sigma$ with finite measure such that $E_\lambda = \bigcup_n G_n$. (Hint: use the functions $h_n = \inf_{m \geq n} g_m$.)
- (b) (10P) Let (Ω, Σ, μ) be a finite measure space. Show that every measurable function is measurable in the strong sense. (Hint: use the functions f_ε below).
- (3) (20P) Let (Ω, Σ, μ) be a measure space such that $\mu(\Omega) < \infty$. Let $f \geq 0$ and $\varepsilon > 0$. Consider

$$f_\varepsilon = \sum_{k=0}^{\infty} (k\varepsilon) 1_{\{k\varepsilon \leq f < (k+1)\varepsilon\}}.$$

Show that

$$I(f) = \lim_{\varepsilon \rightarrow 0} I(f_\varepsilon).$$

Conclude that

$$I(f) = \inf \left\{ \sum_k r_k \mu(E_k) : f \leq \sum_k r_k 1_{E_k} \right\}.$$

- (4) In this exercise we want to establish the link between areas and integrals. Let (Ω, Σ, μ) be a finite measure space ($\mu(\Omega) < \infty$). On $\tilde{\Omega} = [0, \infty) \times \Omega$ we consider the algebra A generated by the sets $[s, t) \times E$, $0 \leq s < t < \infty$, $E \in \Sigma$.

(a) (10P) Show that every element F in A can be written as

$$F = \bigcup_{k=1}^m G_k \times E_k$$

such that the E_k 's are disjoint and $G_k \in A_{\mathbb{R}}$. we then define

$$\nu(F) = \sum_k m(G_k)\mu(E_k).$$

It can be shown that for every $F \in A$ and for every disjoint union $F = \bigcup_j F_j$ of disjoint sets in A we have

$$\nu(F) = \sum_j \nu(F_j).$$

Therefore we may extend ν to a σ -additive measure on the σ -algebras $\tilde{\Sigma}$ of measurable subsets (in the Caratheodory sense) of $[0, \infty] \times \Omega$.

(b) (5P) Let $E \in \Sigma$ be a set of measure 0. Show that

$$\nu([0, \infty) \times E) = 0.$$

(c) (15P) Let $f : \Omega \rightarrow [0, \infty]$ be measurable such that $I(f) < \infty$. Show that the *graph of f*

$$G(f) = \{(r, \omega) : 0 \leq r \leq f(\omega)\}$$

has finite measure with respect to ν and satisfies $\nu(G(f)) \leq I(f)$.

(Hint for a simple function h we have $\nu(G(h)) = I(h)$.)

(d) (10P) Let $f : \Omega \rightarrow [0, \infty]$ be measurable and assume

$$\nu(G(f)) < \infty.$$

Show that $I(f) \leq \nu(G(f))$. (Hint: consider a simple function $0 \leq h \leq f$.)