1. The space of integrable functions

In the following (Ω, Σ, μ) is a measure space.

Definition 1.1. *$f \in S(\mu)$ if $f : \Omega \to \mathbb{R}$ is a measurable function such that $f(\Omega)$ is a finite set and*

$$\mu(f \neq 0) < \infty \, .$$

Then

$$I(f) \; = \sum_{0 \neq r \in f(\Omega)} r\mu(\{f = r\}) \, .$$

Proposition 1.2. *$I : S(\mu) \to \mathbb{R}$ is linear. Moreover, $f \le g$ implies $I(f) \le I(g)$. Furthermore, we have the triangle inequality*

$$I(|f-g|) \le I(|f-h|) + I(|h-g|) \, .$$

Proof. 1. Note that $I(\lambda g) = \lambda I(g)$. Now if $f, g \in S(\mu)$, then

$$f = \sum_{i=0}^{n} x_i 1_{E_i} \text{ where } E_i = f^{-1}(\{x_i\})$$
$$g = \sum_{i=0}^{n} y_i 1_{F_i} \text{ where } F_i = f^{-1}(\{y_i\}),$$

where $x_0 = y_0 = 0$. We may assume that

$$i \neq k \Rightarrow E_i \cap E_k = \emptyset \quad \cup E_i = \Omega \, .$$

Moreover, for $i \neq 0$ we know that $\mu(E_i) < \infty$. Similarly, we may assume

$$j \neq l \Rightarrow F_j \cap F_l = \emptyset \quad \cup F_j = \Omega \, .$$

Consider

$$\{x_i + y_j : 0 \le i \le n, 0 \le j \le m\} \; = \; \{Z_0, Z_1, \dots, Z_N\} \, .$$

We assume that $Z_0 = 0$. Note that $E_0 \cap F_0 \subset (f+g)^{-1}(0)$. Then

$$\begin{aligned} I(f+g) &= \sum_{r=1}^{N} Z_r \mu(\cup_{x_i+y_j=Z_r} E_i \cap F_j) \\ &= \sum_{i,j,\min(i,j)>0} (x_i + y_j)\mu(E_i \cap F_j) \\ &= \sum_{i=1}^{n} x_i \sum_{j=0}^{m} \mu(E_i \cap F_j) + \sum_{j=1}^{m} y_j \sum_{i=0}^{n} \mu(E_i \cap F_j) \\ &= \sum_i x_i \mu(E_i) + \sum_j y_j \mu(F_j) \; = \; I(f) + I(g) \, . \end{aligned}$$

2. If $f \le g$, then $E_i \cap F_j \neq \emptyset$ implies $x_i \le y_j$. Let $G = \bigcup_{i=1}^n E_i)$. This yields

$$\begin{aligned} I(f) = \sum_{i=1}^n x_i \mu(E_i) \ &= \ \sum_{i=1}^n x_i \sum_{j=0, E_i \cap F_j \neq \emptyset}^m \mu(E_i \cap F_j) \\ &\le \sum_{i>1 E_i \cap F_j \neq \emptyset} y_j \mu(E_i \cap F_j) \ = \ \sum_{j=0}^m y_j \sum_{i=1}^m \mu(E_i \cap F_j) \\ &= \sum_{j=1}^m y_j \sum_{i=1}^n \mu(E_i \cap F_j) \ = \ \sum_{j=1}^m y_j \mu(F_j \cap G) \\ &= I(g 1_G) \, . \end{aligned}$$

However, if $\omega \in G^c$, then $f(\omega) = 0$ and hence $g(\omega) \ge 0$. This yields

$$I(g 1_{G^c}) \ \ge \ 0 \, .$$

Therefore $I(f) \le I(g1_G) \le I(g1_G) + I(g1_{G^c}) \ = \ I(g)$.
3. We note $f - g \le |f - g$ and hence

$$I(f) - I(g) \ = \ |I(f-g)| \le I(|f-g|) \, .$$

Similarly, $I(g) - I(f) \le I(|g-f|) \ = \ I(|f-g|)$. The assertion follows. ∎

Definition 1.3. *Let $f : \Omega \to [0, \infty]$ a positive measurable function. Then*

$$I(f) \ = \sup_{0 \le h \le f, h \in S(\mu)} I(h) \, .$$

f is called (positive) integrable if $I(f)$ is finite.

Lemma 1.4. *1) $f \le g$, then $I(f) \le I(g)$. 2) $I(f+g) \ \ge \ I(f) + I(g)$*

Proof. 2) Let $0 \le f_n \le f$ and $0 \le g_n \le g \Rightarrow 0 \le f_n + g_n \le f + g$. Then,

$$I(f) + I(g) = \sup I(f_n) + \sup I(g_n) \le I(f+g) \, .$$

∎

We want to show that for integrable f, g we have $I(f+g) \le I(f) + I(g)$. This will be done in several steps.

Lemma 1.5. *$f \ge 0$ integrable, $0 \le h \le f$, $h \in S(\mu)$. Then*

$$I(f) \ = \ I(f-h) + I(h) \, .$$

Proof. Let $\epsilon > 0$ and $0 \le \tilde{g} \le f$, $\tilde{g} \in S(\mu)$ such that

$$I(\tilde{g}) \le I(f) \le I(\tilde{g}) + \epsilon.$$

Define $g = \max\{\tilde{g}, h\} \geq \tilde{g}$, then

$$I(g) \leq I(f) \leq I(\tilde{g}) + \epsilon \leq I(g) + \epsilon = I(g-h) + I(h) + \epsilon \leq I(f-h) + I(h) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this concludes the proof. ■

LEMMA 1.6. *$f \geq 0$ integrable. Then there exits a sequence $f_n \in S(\mu)$ such that $0 \leq f_n \leq f$, f_n is increasing and*

$$I(f - f_n) \ \leq \ 4^{-n} \ .$$

PROOF. Let $\epsilon = 2^{-n}$. Furthermore, let $0 \leq \tilde{g}_n \leq f$ and $I(f) \leq I(\tilde{g}_n) + \epsilon_n$. Define $g_n = \max\{\tilde{g}_1, \dots, \tilde{g}_n\}$, then

$$I(g_n) \leq I(f) \leq I(\tilde{g}_n) + \epsilon_n \leq I(g_n) + \epsilon_n.$$

By Lemma 1.5:

$$I(g_n) + \epsilon_n \geq I(f) = I(f - g_n) + I(g_n) \ .$$

Subtracting $I(g_n)$ yields $I(f - g_n) < \varepsilon_n$. ■

LEMMA 1.7. *(Chebychev) f integrable, $\lambda > 0$. Then*

$$\lambda \mu(f \geq \lambda) \ \leq \ I(f) \ .$$

PROOF. Let $h = \lambda 1_{f \geq \lambda} \leq f$ and let $\Omega_n \subset \Omega$ an increasing sequences of subsets such that $\cup \Omega_n = \Omega$. Then

$$h_n = \lambda 1_{\{f \geq \lambda\} \cap \Omega_n} \leq I(f),$$

so

$$\mu(f \geq \lambda) = \lim_n \mu(\{f \geq \lambda\} \cap \Omega_n) = \lim_n \frac{1}{\lambda} I(h_n) \leq \frac{1}{\lambda} I(f) \ .$$ ■

LEMMA 1.8. *Let (f_n) and f be measurable such that*

$$I(|f_n - f|) \ \leq \ 4^{-n}$$

Then f_n converges to f almost everywhere.

PROOF. Let $E_n = \{\omega : |f - f_n| > 2^{-n}\}$, $F_n = \cup_{k \geq n} E_k$ and $F = \cap_n F_n$, then

$$\mu(F) \leq \lim_n \mu(F_n) \leq \lim_n \sum_{k \geq n} \mu(E_k) = \lim \sum_{k \geq n} 2^{-k} = \lim 2^{-n} = 0.$$

If $\omega \notin F \Rightarrow \exists n$ such that $\forall \ k \geq n$, $f(\omega) - 2^{-k} \leq f_k(\omega) \leq f(\omega) + 2^{-k}$, which implies

$$\lim_k f_k(\omega) = f(\omega) \ .$$ ■

Lemma 1.9. *(Beppo-Levi) $0 \le f_n \le f_{n+1}$ all integrable such that $\lim_n I(f_n) < \infty$. If f_n converges to f alomst everywhere, then*

$$I(f) \ \le \ \lim_n I(f_n)\,.$$

Proof. Let $0 \le h \le f$, $h \in S(\mu)$ and

$$h = \sum_{i=1}^m r_i 1_{E_i}$$

Let $F \subset \Omega$ such that $\lim f_n(\omega) = f(\omega)\ \forall\ \omega \in F$. Given any $\epsilon > 0$, define

$$E_{i,n} = \left\{\omega \in E_i \cap F : f_n(\omega) > \frac{r_i}{1+\epsilon}\right\} \subset E_i,$$

then $E_i \cap F = \cup_n E_{i,n}$ because $\lim_n f_n(\omega) = f(\omega) \ge r_i$. We may find n such that $\mu(E_{i,n}) \ge \frac{\mu(E_i \cap F)}{1+\epsilon}$ for all $i = 1, \dots, m$.
Then define

$$h^{\epsilon,n} = \sum_i \frac{r_i}{1+\epsilon} 1_{E_{i,n}} \le f_n 1_F \le f_n\,.$$

Note that $\le f_n 1_F \le f_n$ and hence

$$I(h^{\epsilon,n}) \le I(f_n) \le \lim_n I(f_n)\,.$$

Moreover,

$$\begin{aligned} I(h) &= \sum_{i=1}^m r_i \mu(E_i) = \sum_{i=1}^m r_i \mu(E_i \cap F) \\ &\le (1+\epsilon)^2 \sum_{i=1}^m \frac{r_i}{1+\epsilon} \mu(E_{i,n}) \\ &\le (1+\epsilon)^2 I(h^{\epsilon,n}) \le (1+\epsilon)^2 I(f_n) \ \le \ (1+\varepsilon)^2 \lim_n I(f_n)\,. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $I(h) \le \lim_n I(f_n)$. Taking the supremum, we deduce the assertion. ■

Corollary 1.10. *(Fatou) Let $0 \le f_n$ be positive integrable functions. Then*

$$I(\liminf_n f_n) \ \le \ \liminf_n I(f_n)\,.$$

Proof. The sequence $g_n = \inf_{m \ge n} f_m$ is increasing. Then

$$I(\sup_n g_n) = \sup_n I(g_n) \ \le \ \sup_n \inf_{m \ge n} I(f_m) \ = \ \liminf_n I(f_n)\,.$$ ■

Proposition 1.11. *f, g positive integrable. Then*

$$I(f) + I(g) \ = \ I(f+g)\,.$$

PROOF. Let $f_n \in S(\mu)$, $g_n \in S(\mu)$ increasing sequences such that $I(f-f_n) \le 4^{-n}$ and $I(g-g_n) \le 4^{-n}$. Then the sequence $h_n = f_n + g_n$ converges to $f+g$ almost everywhere and

$$I(f+g) \le \lim_n I(f_n+g_n) \;=\; \lim_n I(f_n) + I(g_n) \;=\; I(f)+I(g) \,. \qquad \blacksquare$$

DEFINITION 1.12. *A measurable function $f : \Omega \to [-\infty,\infty]$ is called integrable if there exists a sequence (f_n) in $S(\mu)$ such that*

$$\lim_n I(|f-f_n|) \;=\; 0 \,.$$

We denote $I(\mu)$ the space of integrable functions

PROPOSITION 1.13. *Let f be μ-integrable and (f_n), (f_n') such that*

$$\lim_n I(|f-f_n|) \;=\; 0 \;=\; \lim_n I(|f-f_n'|)$$

Then

$$\lim_n I(|f_n-f_n'|) \;=\; 0 \,.$$

In particular,

$$\int f \;=\; \lim_n I(f_n)$$

is well-defined.

LEMMA 1.14. *Let $f \ge 0$ be μ-integrable. Then*

$$I(f) \;=\; \int f \,.$$

PROOF. Let (f_n) be a sequence of simple functions such that $I(|f-f_n|) \le 4^{-n}$. Then f_n converges to f almost everywhere. For fixed $n \in \mathbb{N}$ we consider $E_n = \{\omega : f_n(\omega) < 0$. Then

$$1_{E_n}|f-f_n| \;=\; 1_{E_n}|f| + 1_{E_n}|f_n| \;\ge\; 1_{E_n}|f| \,.$$

Thus we have $I(|f-f_n^+|) \;\le\; I(|f-f_n|) \;\le\; 4^{-n}$. Then f_n^+ converges to f μ-almost everywhere and Fatou's lemma implies

$$I(f) \;\le\; \liminf_n I(f_n^+) \;\le\; \sup_n I(|f_n|) \,.$$

However,

$$\begin{aligned} |I(|f_n|) - I(|f_m|)| = |I(|f_n| - |f_m|)| \;&\le\; I(||f_n| - |f_m||) \;\le\; I(|f_n - f_m|) \\ &\le I(|f_n - f|) + I(|f - f_m|) \;\le\; 4^{-n} + 4^{-m} \,. \end{aligned}$$

Thus $I(|f_n|)$ is Cauchy and hence $\sup_n I(|f_n|)$ bounded. We get

$$I(h) \le \sup_n I(|f_n|) .$$

In particular, $I(f)$ is finite. Equality follows from the preceding Proposition. ∎

PROPOSITION 1.15. *1) $f, g \in I(\mu)$, $\lambda \in \mathbb{R}$. Then $\int (f + \lambda g) = \int f + \lambda \int g$.*
2) $f, g \in I(\mu)$, $f \le g$. Then $I(f) \le I(g)$.

PROOF. 1) There exist $I(|f_n - f|) \to 0$ and $I(|g_n - g|) \to 0$, then

$$I(|f_n + \lambda g_n - (f + \lambda g)|) \le I(|f_n - f|) + \lambda I(|g_n - g|) \to 0.$$

Since integral is well defined, so $\int f + \lambda g = \lim_n I(f_n + \lambda g_n) = I(f) + \lambda g$.
2) Consider $h = f - g \ge 0$, then $\int h = I(h) \ge 0$. Thus

$$\int f = \int (f - g + g) = \int h + \int g \ge \int g .$$ ∎

PROPOSITION 1.16. *f is integrable iff f^+ and f^- are integrable. Moreover, we have*

$$\int f = \int f^+ - \int f^- .$$

PROPOSITION 1.17. *f is integrable iff f^+ and f^- are integrable. Moreover, we have*

$$\int f = \int f^+ - \int f^- .$$

PROOF. "$\Leftarrow$": $f = f^+ - f^-$.
"$\Rightarrow$": There exists f_n such that

$$\lim I(|f - f_n|) = 0 \Rightarrow \lim_n I(||f| - |f_n||) \Rightarrow 0,$$

so $|f|$ is integrable.

$$\int f = \int \frac{|f| + f}{2} + \int \frac{f - |f|}{2} = \int f^+ - \int f^- .$$ ∎

2. Convergence Theorems and applications

LEMMA 2.1. *(Fatou) Let (f_n) be positive integrable functions. Then*

$$\int \liminf_n f_n \le \liminf_n \int f_n .$$

Theorem 2.2. *(Dominated convergence theorems) Let $f \geq 0$ be positive integrable function. Let (g_n) be integrable functions such that*

$$|g_n| \leq f \quad \mu\, a.e$$

for all $n \in \mathbb{N}$ and $g = \lim_n g_n$ exists. Then g is integrable and

$$\int g \;=\; \lim_n \int g_n \, .$$

Proof. Let us first assume $|g_n| \leq f$ everywhere and $g = \lim_n g_n$. Then the sequence $h_n = g_n + f$ is positive. By Fatou's Lemma, we find

$$\int g + f \;=\; \int \lim_n g_n + f \;\leq\; \liminf_n \int g_n + \int f \, .$$

Thus $g + f$ and hence g is integrable. Subtracting $\int f$ we get

$$\int g \;\leq\; \liminf_n \int g_n \, .$$

Now, we consider $k_n = -g_n + f$ and deduce similarly as before

$$-\int g + \int f \;\leq\; \liminf_n \int -g_n + \int f \, .$$

Thus

$$\limsup_n \int g_n \;\leq\; \int g \, .$$

In the general case, we consider the exceptional set $E_n \in \Sigma$ of measure 0 such that $|g_n(\omega)| \leq f(\omega)$ for all $\omega \in E_n^c$. Let $F \in \Sigma$ be of measure 0 such that $\lim_n g_n(\omega) = g(\omega)$ holds for $\omega \in F^c$. We define $E = F \cup \bigcup_n E_n$. Then, we have $\mu(E) = 0$. Moreover, the functions $\tilde{g_n} = g_n 1_{E^c}$ and $\tilde{g} = g 1_{E^c}$ satisfy all the requirements above. The assertion follows from the following remark. ∎

Remark 2.3. *Let E be a set of measure 0 and $f : \Omega \to 0$ be a measurable function, then $\int |f| 1_E = 0$.*

Proof. Let $0 \leq h \leq |f| 1_E$. We may write

$$h \;=\; \sum_{i=1}^{m} r_i 1_{F_i} \, .$$

Then $h 1_E \;=\; h$ and hence

$$h \;=\; \sum_{i=1}^{m} r_i 1_{E \cap F_i} \, .$$

This yields $I(h) = 0$. ∎

We will now discuss an application.

THEOREM 2.4. *(Riemann-Lebesgue lemma) Let $f : \mathbb{R} \to [-\infty, \infty]$ be integrable. Then*

$$\lim_k \int \cos(kt) f(t) dt \;=\; 0 \,.$$

LEMMA 2.5. *Let f be a step function. Then*

$$\lim_k \int \cos(kt) f(t) dt \;=\; 0 \,.$$

PROOF. Let $f = \sum_{i=1}^m r_i 1_{[a_i,b_i]}$. By linearity it suffices to show that

$$\lim_k \int_a^b \cos(tk) dt \;=\; 0 \,.$$

This follows obviously from

$$|\int_a^b \cos(tk) dt| \;=\; \frac{|\sin(bkt) - \sin(akt)|}{k} \;\le\; \frac{2}{k} \,.$$

The assertion is proved. ∎

The Riemann-Lebsgue lemma is an easy consequence of the following result.

THEOREM 2.6. *Let $f : \mathbb{R} \to [-\infty, \infty]$ be integrable and $\varepsilon > 0$. Then there exits a simple function h such that*

$$\int |f - h| < \varepsilon \,.$$

Proof of Theorem 2.4 from Theorem 2.6. Let h be a simple function with $\int |f - h| < \frac{\varepsilon}{2}$. Let k_0 such that

$$|\int \cos(kt) h(t) dt| < \frac{\varepsilon}{2}$$

for all $k > k_0$. Then

$$\begin{aligned} |\int \cos(kt) f(t) dt| &\le |\int \cos(kt)(f(t) - h(t)) dt| + |\int \cos(kt) h(t) dt| \\ &< \int |f - h| + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

holds for all $k > k_0$. ∎

The following Lemma is an immediate application of the dominated convergence theorem:

LEMMA 2.7. *Let $f : \mathbb{R} \to [-\infty, \infty]$ be integrable and $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$ such that*

$$\int_{|x| \ge n} |f| < \varepsilon \,.$$

Proof of Theorem 2.6. Let $\varepsilon > 0$. We choose $n \in \mathbb{N}$ such that

$$\int_{|x|\geq n} |f| < \frac{\varepsilon}{3} .$$

Let $h : [-n, n] \to \mathbb{R}$ a simple function such that

$$\int_{-n}^{n} |f - h| < \frac{\varepsilon}{3} .$$

Let $C = \sup |h|$ and $\delta = \frac{\varepsilon}{6(C+n)}$. We apply the consequence of Lusin's theorem and find an simple function g such that

$$m(|g - h| > \delta) < \delta .$$

Moreover, the construction yields such an h with $|h| \leq C$. Then, we get

$$\begin{aligned}\int_{-n}^{n} |g - h| &= \int_{-n}^{n} 1_{|g-h|>\delta}|g - h| + \int_{-n}^{n} 1_{|g-h|\leq\delta}|g - h| \\ &\leq 2Cm(|g - h| > \delta) + 2n\delta \ \leq \ 2(C + n)\delta < \frac{\varepsilon}{3} .\end{aligned}$$

We insist that $h = h1_{[-n,n]}$. Thus we get

$$\int |f - h| \leq \int_{|x|>n} |f| + \int_{-n}^{n} |f - h| \ \leq \ \int_{|x|>n} |f| + \int |f - g| + \int_{-n}^{n} |g - h| < \varepsilon. \quad \blacksquare$$

LEMMA 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a step function and $\varepsilon > 0$. Then there exits a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} |g(x)| = 0$ and*

$$\int |f - g| < \varepsilon .$$

PROOF. Let $f = 1_{[a,b]}$ and $0 < \delta < b - a$. Then

$$g_{\delta,a,b}(t) \ = \ \begin{cases} \delta^{-1}(t - (a - \delta)) & \text{if } a - \delta \leq t \leq a \\ 1 & \text{if } a \leq t \leq b \\ 1 - \delta^{-1}(t - b) & \text{if } b \leq t \leq b + \delta \\ 0 & \text{else} \end{cases} .$$

Then g is continuous and

$$\int |g_{\delta,a,b} - 1_{[a,b]}|d\mu \leq 2 \int_{0}^{\delta} tdt \ = \ \delta .$$

For an arbitrary simple function $f = \sum_{i=1}^n r_i 1_{[a_i,b_i]}$ we consider $g_\delta = \sum_{i=1}^n r_i g_{\delta,a_i,b_i}$. Then we get

$$\int |f - g_\delta| \leq \sum_{i=1}^n |r_i| \int |1_{[a,b_i]} - g_{\delta,a_i,b_i}(t)| \ \leq \ \sum_{i=1}^n |r_i|\delta \ .$$

Thus $\delta < \frac{\varepsilon}{1+\sum_{i=1}^n |r_i}$ implies the assertion. ■

COROLLARY 2.9. *Let $f : \mathbb{R} \to [-\infty, \infty]$ be an integrabe function and $\varepsilon > 0$. Then there exits a continuous function g vanishing at $\pm\infty$ such that*

$$\int |f - g| d\mu < \varepsilon \ .$$

PROOF. Let h be a step function such that

$$\int |f - h| d\mu < \frac{\varepsilon}{2} \ .$$

Let g be a continuous function (constructed above) such that $\int |g - h| < \frac{\varepsilon}{2}$. This function g vanishes for large x's and satisfies

$$\int |f - g| d\mu \ \leq \ \int |f - h| d\mu + \int |h - g| < \varepsilon \ .$$

This proves the assertion. ■