December 19

Final (441 Section)

(1) (a) (20P) Let $f \in L_\infty([0,1])$. Show that

$$\|f\|_\infty \;=\; \sup_{p<\infty} \|f\|_p \;.$$

(b) (20P) Recall that a non-decreasing function $f : [0,1] \to \mathbb{R}$ of bounded variation is singular, if $f' = 0$ a.e. Show that for every $\varepsilon > 0$ and $\delta > 0$ there exists $0 < x_1 < y_1 < x_2 < y_2 < x_3 \cdots < x_n < y_n < 1$ such that

$$\sum_{k=1}^{n} y_k - x_k < \delta$$

but

$$\sum_{k=1}^{n} f(y_k) - f(x_k) > f(1) - f(0) - \varepsilon \,.$$

Hint apply the Vitali covering Lemma for $\{[y, y+h] \mid f(y+h) - f(h) < \varepsilon h, y \in E\}$ with $m(E^c) = 1$.

(c) Let m_c be the counting measure on $\mathbb{N}$ and $\ell_p \;=\; L_p(\mathbb{N}, m_c)$.

(i) (10P) Let $1 \le p \le q \le \infty$. Show

$$\ell_p \subset \ell_q \,.$$

(ii) (20P) Is it true that for $1 \le p \le \infty$ the simple functions are dense in ℓ_p?

(iii) (20P) Is it true that for all $1 \le p \le \infty$ and $f : \mathbb{N} \to \mathbb{R}$

$$\|f\|_p \;=\; \sup \|f_n\|_p \;,$$

$$\text{where } f_n(k) = \begin{cases} f(k) & \text{if } k \leq n \\ 0 & \text{else} \end{cases}$$

(iv) (30P) Show that $\ell_p^* = \ell_{p'}$ for all $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

(v) (20P) Show that on $\mathbb{N} \times \mathbb{N}$ the product measure $m_c \times m_c$ coincides with the counting measure.

(d) (30P) Let $f : [0,1] \to \mathbb{R}_{\geq 0}$ be in $L_1([0,1])$. Show that

$$\int_0^1 f(t) dm(t) \;=\; m_2(\{(x,y) \mid 0 \leq x \leq 1\ ,\ 0 \leq y \leq f(x)\})\ .$$

In which sense is the right hand side well-defined?