Practice problems

(1) ('Easy question') Let $r_k > 0$ and $E_k \in \Sigma$ disjoint sets with finite measure. We assume that

$$\sum_k r_k \mu(E_k) \; < \; \infty$$

Show that $f \; = \; \sum_k r_k 1_{E_k}$ is integrable and satisfies

$$I(f) \; = \; \sum_k r_k \mu(E_k) \, .$$

Hint: Use Fatou (actually Beppo Levi) and dominated convergence theorem.

(2) Let f be an integrable function on $\mathbb{R}$ and g be a bounded measurable function. Show that

$$\lim_{t \to 0} \int |g(x)(f(x+t) - f(x))| dx \; = \; 0 \, .$$

Hint: First show this for f continuous and $f(x) = 0$ for $|x| \geq n$. Now use the fact that every integrable function f can be approximated by a continuous function h such that $\int |f - h| < \frac{\varepsilon}{3}$. You may also use that $\int f(x+t) dx \; = \; \int f(x) dx$.

(3) Don't look at the notes and show

(a) The for a σ-finite measure space and a positive function f with $\int |f|^2 < \infty$ you can find an increasing sequence of simple functions $h_n \leq f$ such that

$$\int (|f|^2 - |h_n|^2) \; \leq \; 4^{-n} \, .$$

Conclude that $\mu(f^2 - h_n^2 > 2^{-n}) < 2^{-n}$. Thus h_n^2 converges to f^2 and henceforth h_n converges to f a.e. Use this to show

$$\lim_n \int |f - h_n|^2 \; = \; 0 \, .$$

(b) Look at the web for the notes on the Lusin theorem and its consequences. Show that for every function $f : [-N, N] \to \mathbb{R}$ with $\int_{-N}^{N} |f|^2 < \infty$ and $\varepsilon > 0$ there exists a continuous function g such that

$$\int_{-N}^{N} |f - g|^2 < \varepsilon \, .$$

(4) Let (g_n) be a sequence positive integrable functions and (f_n) and integrable sequence such that $|f_n| \leq g_n$. We assume that f_n converges to f and

$$\lim_n \int g_n = \int g$$

Show that

$$\lim_n \int f_n = \int f .$$

(Remark: After the fact the argument can easily modified to the situation where a.e. is added in all the relevant places.)

(5) We will now discuss the metric associated to 'convergence in measure'. Let L_0 be the set of equivalence classes of measurable functions satisfying $\lim_{\alpha \to \infty} \mu(|f| > \lambda) = 0$.

(a) Show that

$$d([f],[g]) = \inf\{\varepsilon : \mu(|f-g| > \varepsilon) < \varepsilon\}$$

satisfies the triangle inequality.

(b) Show that if $([f_n])$ is Cauchy with respect to d, then there exists a subsequence $([f_{n_k}])$ such that

$$\mu(|f_{n_{k+1}} - f_{n_k}| > 2^{-k}) < 2^{-k} .$$

In this case (f_{n_k}) converges a.e.

(c) Show that (L_0, d) is a complete metric space.

(6) Let $\alpha > 0$. On the space vector space V of finite sequences

$$V = \{(a_n) : \exists_{n_0} \forall_{n > n_0} a_n = 0\}$$

we define the norm

$$\|(a_n)\| = \sum_n e^{\alpha n} |a_n|$$

Show that every continuous linear functional $\phi : (V, \| \|) \to \mathbb{R}$ is given by a sequence (x_n) and

$$\phi_{(x_n)}((a_n)) = \sum_n a_n x_n \quad \text{and} \quad \|\phi_{(x_n)}\| = \sup_n e^{-\alpha n} |x_n| .$$

Remark: For ODE the modified norms on $C(\mathbb{R})$

$$\|f\| = \sup_t e^{-\alpha |t|} |f(t)|$$

are important. Above you see a discrete analogue of this norm.