

Solutions for the practice problems

1) ('Easy question') Let $r_k > 0$ and $E_k \in \Sigma$ disjoint sets with finite measure. We assume that

$$\sum_k r_k \mu(E_k) < \infty$$

Show that

$$f = \sum_k r_k 1_{E_k}$$

is integrable and satisfies

$$I(f) = \sum_k r_k \mu(E_k).$$

Hint: Use Fatou and dominated convergence theorem.

Proof: By Beppo Levi

$$\begin{aligned} \int \sum_k r_k 1_{E_k} &= \int \sup_n \sum_{k \leq n} r_k 1_{E_k} \leq \sup_n \int \sum_{k \leq n} r_k 1_{E_k} = \sup_n \sum_k r_k \mu(E_k) \\ &= \sum_k r_k \mu(E_k). \end{aligned}$$

Hence $f = \sum_k r_k 1_{E_k}$ is integrable and we may apply the DCT for $f_n = \sum_{k \leq n} r_k 1_{E_k}$ and $f = \lim_n f_n$ and majorant f .

Let f be an integrable function on \mathbb{R} and g be a bounded measurable function. Show that

$$\lim_{t \rightarrow 0} \int |g(x)(f(x+t) - f(x))| = 0.$$

Hint: First show this for f continuous and $f(x) = 0$ for $|x| \geq n$. Now use the fact that every integrable function f can be approximated by a continuous function h such that $\int |f - h| < \frac{\epsilon}{3}$.

Proof: Let us assume that f continuous and $f(x) = 0$ for $|x| \geq n$. Let (t_k) be an arbitrary sequence converging to 0 such that $|t_k| \leq 1$. Then

$$\lim_k |g(x)(f(x+t_k) - f(x))| = 0$$

for all $|x| < n$. Since $f : [-n, n] \rightarrow \mathbb{R}$ is continuous it is also bounded. Let us define $C = \sup |f|$ and

$$h = \sup |g| 2C 1_{[-(n+1), n+1]}.$$

Given $x \in \mathbb{R}$ and $0 \leq |t| \leq 1$ we observe that

$$|g(x)(f(x+t) - f(x))| \leq |g(x)| |f(x+t)| + |f(x)| \leq h(x)$$

because $x + t \in [-(n+1), (n+1)]$. Thus the dominated convergence theorem yields

$$\lim_k \int |g(x)(f(x + t_k) - f(x))| = \int \lim_k |g(x)(f(x + t_k) - f(x))| = 0.$$

Since (t_k) is arbitrary we deduce

$$\lim_{t \rightarrow 0} \int |g(x)(f(x + t) - f(x))|.$$

Now, let f be an arbitrary function and h be a continuous function with finite support such that $\int |f - h| < \frac{\varepsilon}{3 \sup |g|}$ (see the notes on Lusin's theorem now on the web). We choose t_0 such that $|t| < t_0$ implies

$$\int |g(x)(h(x + t) - h(x))| < \frac{\varepsilon}{3}.$$

Then we get

$$\begin{aligned} & \int |g(x)(f(x + t) - f(x))| \\ & \leq \int |g(x)(f(x + t) - h(x + t))| + \int |g(x)(h(x + t) - h(x))| + \int |g(x)(h(x) - f(x))| \\ & < \sup |g| \int |f(x + t) - h(x + t)| + \frac{\varepsilon}{3} + \sup |g| \int |(h(x) - f(x))| \\ & < \sup |g| \int |f(x + t) - h(x + t)| + \frac{2}{3}\varepsilon < \varepsilon. \end{aligned}$$

In the last line we used

$$(0.1) \quad \int u(x + t) dm(x) = \int u(x) dm(x)$$

for arbitrary integrable functions. This equality (0.1) is proved similarly. First we observe that it is true for step functions (by the translation invariance of the Lebesgue measure (very easy for step functions)). Thus the linear map

$$T_t(f)(x) = f(x + t)$$

is defined on a dense subset L_1 with values in L_1 and satisfies

$$\|T_t(f)\|_1 = \|f\|_1$$

on a dense set. The unique extension principle allows us to extend T_t to a continuous linear map \tilde{T}_t on $L_1(\mathbb{R})$ still satisfying

$$\|T_t(f)\|_1 = \|f\|_1.$$

We also have a Lipschitz map $I : L_1(\mathbb{R}) \rightarrow R$ given by

$$I([f]) = \int f(x)dm(x).$$

Note that now for sequence (h_n) converging to f we have

$$I(\tilde{T}_t([f])) = \lim_n I([T_t(h_n)]) = \lim_n I([h_n]) = I([f]) = \int f(x)dm(x).$$

Funny enough, it requires extra work to conclude that for $g \in \tilde{T}_t([f])$ we have $g = T_t(f)$ almost everywhere. Indeed, let (h_n) be a sequence of step functions which converges to f such that $\|f - h_n\| \leq 4^{-n}$. Then we may assume that (h_n) converges to f almost everywhere. Let F be a set of measure 0 such that $f(x) = \lim_n h_n(x)$ for all $x \in F^c$. By the definition of the outer measure we know that $F - t = \{x + t : x \in F\}$ has also measure 0. Let $x \in F^c \cap (F - t)^c$. Then $x + t \notin F$ and hence

$$\lim_n h_n(x) = f(x)$$

and

$$\lim_n h_n(x + t) = f(x + t).$$

Thus $T_t(h_n)$ converges to $T_t(f)$ almost every where. Finally $\|T_t(h_n) - \tilde{T}_t(f)\|_1 = \|h_n - f\|_1 \leq 4^{-n}$ guarantees that $T_t(h_n)$ converges to g almost everywhere. Thus $g = T_t(f)$ holds almost everywhere and hence

$$\int f(x + t)dm(x) = \int_{(F \cap F - t)^c} f(x + t)dm(x) = \int g(x)dm(x) = \int f(x)dm(x).$$

Don't look at the notes and show

- (1) For a σ -finite measure space and a positive function f with $\int |f|^2 < \infty$ you can find an increasing sequence of simple functions $h_n \leq f$ such that

$$\int (|f|^2 - |h_n|^2) \leq 4^{-n}.$$

Conclude that $\mu(f^2 - h_n^2) > 2^{-n} < 2^{-n}$. Thus h_n^2 converges to f^2 and henceforth h_n converges to f a.e. Use this to show

$$\lim_n \int |f - h_n|^2 = 0.$$

- (2) Look at the web for the notes on the Lusin theorem and its consequences. Show that for every function $f : [-m, m] \rightarrow \mathbb{R}$ with $\int |f|^2 dm < \infty$ and $\varepsilon > 0$ there exists a continuous function g such that

$$\int |f - g|^2 dm < \varepsilon.$$

Solution: See notes.

2) Let (g_n) be a sequence positive integrable functions and (f_n) and integrable sequence such that $|f_n| \leq g_n$. We assume that f_n converges to f , g_n converges to g and

$$\lim_n \int g_n = \int g$$

Show that

$$\lim_n \int f_n = \int f.$$

(Remark: After the fact the argument can easily modified to the situation where a.e. is added in all the relevant places.)

Proof: Define $h_n = f_n + g_n$ which is positive. By Fatou

$$\int f + g = \int \liminf_n h_n \leq \liminf_n \int f_n + \int g_n = \liminf_n \int f_n + \int g,$$

because $\lim_n \int g_n = \int g$. Subtracting $\int g$ yields

$$\int f \leq \liminf_n \int -f_n.$$

Apply the same for $k_n = -f_n + g_n$ and we get

$$\int -f \leq \liminf_n \int -f_n.$$

Thus

$$\limsup_n \int f_n \leq \int f \leq \liminf_n \int f_n.$$

Thats it. ■

3) We will now discuss the metric associated to ‘convergence in measure’. Let L_0 be the set of equivalence classes of measurable functions satisfying $\lim_{\alpha \rightarrow \infty} \mu(|f| > \lambda) = 0$.

(1) Show that

$$d([f], [g]) = \inf\{\varepsilon : \mu(|f - g| > \varepsilon) < \varepsilon\}$$

satisfies the triangle inequality.

Proof: Let h be a further function and $d([f], [h]) < \varepsilon$, $d([f], [h]) < \delta$ then

$$\mu(|f - h| > \varepsilon) < \varepsilon \quad \text{and} \quad \mu(|h - g| > \delta) < \delta.$$

Note that

$$\begin{aligned} \{\omega : |f(\omega) - g(\omega)| > \varepsilon + \delta\} &\subset \{\omega : |f(\omega) - h(\omega)| > \varepsilon \text{ or } |h(\omega) - g(\omega)| > \delta\} \\ &\subset \{\omega : |f(\omega) - h(\omega)| > \varepsilon\} \cup \{\omega : |h(\omega) - g(\omega)| > \delta\} \end{aligned}$$

because $|f(\omega) - h(\omega)| \leq \varepsilon$ and $|h(\omega) - g(\omega)| \leq \delta$ implies $|f(\omega) - g(\omega)| \leq \varepsilon + \delta$. Thus we have

$$\mu(|f - g| > \varepsilon + \delta) < \varepsilon + \delta.$$

Hence

$$d([f], [g]) \leq \varepsilon + \delta.$$

Taking the infimum yields the assertion. ■

(2) Show that if $([f_n])$ is Cauchy with respect to d , then there exists a subsequence $([f_{n_k}])$ such that

$$(0.2) \quad \mu(|f_{n_{k+1}} - f_{n_k}| > 2^{-k}) < 2^{-k}.$$

In this case (f_{n_k}) converges a.e.

Proof: We can always pass to a subsequence such that

$$d([f_{n_{k+1}}], [f_{n_k}]) < 2^{-k}.$$

Thus (0.2) holds. We follow the standard trick

$$E_j = \{\omega : |f_{n_{j+1}} - f_{n_j}| > 2^{-j}\}$$

and

$$F_k = \bigcup_{j \geq k} E_j$$

Then $\mu(F_k) \leq 2^{1-k}$ and hence $F = \bigcup_k F_k$ has measure 0. For $\omega \in F^c$ we can find k such that for all $j \geq k$

$$|f_{n_{j+1}}(\omega) - f_{n_j}(\omega)| \leq 2^{-j}.$$

Thus $f(\omega) = \lim_j f_{n_j}(\omega)$ exists on F^c . ■

(3) Show that (L_0, d) is a complete metric space.

Proof: It suffices to show that for every sequence $([f_n])$ satisfying $d([f_{n+1}], [f_n]) < 2^{-n}$ has a limit. By the argument above, we know that (f_n) converges a.e. for a limit f . Moreover, we use

$$F_n = \bigcup_{j \geq n} \{\omega : |f_{j+1}(\omega) - f_j(\omega)| > 2^{-j}\}.$$

Then $\mu(F_n) \leq 2^{1-n}$. For $\omega \in F_n^c$ we know that $f(\omega) = \lim_j f_j(\omega)$ converges and

$$|f(\omega) - f_n(\omega)| = \left| \sum_{j=n}^{\infty} f_{j+1}(\omega) - f_j(\omega) \right| \leq \sum_{j=n}^{\infty} |f_{j+1}(\omega) - f_j(\omega)| \leq 2^{1-n}.$$

Let $\varepsilon > 0$ then

$$\mu(|f - f_n| > (1 + \varepsilon)2^{1-n}) \leq 2^{1-n} < (1 + \varepsilon)2^{1-n}.$$

This shows that

$$d([f_n], [f]) \leq 2^{1-n}.$$

That's enough. ■

4) Let $\alpha > 0$. On the space vector space V of finite sequences

$$V = \{(a_n) : \exists_{n_0} \forall_{n > n_0} a_n = 0\}$$

we define the norm

$$\|(a_n)\| = \sum_n e^{\alpha n} |a_n|$$

Show that every continuous linear functional $\phi : (V, \|\cdot\|) \rightarrow \mathbb{R}$ is given by a sequence (x_n) and

$$\phi_{(x_n)}((a_n)) = \sum_n a_n x_n$$

and

$$\|\phi_{(x_n)}\| = \sup_n e^{-\alpha n} |x_n|.$$

Remark: For ODE the modified norms on $C(\mathbb{R})$

$$\|f\| = \sup_t e^{-\alpha|t|} |f(t)|$$

are important. Above you see a discrete analogue of this norm.

Proof: Let $\phi : V \rightarrow \mathbb{R}$ be a continuous linear functional of $\|\phi\| \leq 1$. Then we may define

$$x_n = \phi(e_n).$$

Here $e_n = (0 \cdots 0, \underbrace{1}_{n\text{-the position}}, 0 \cdots)$ is the n -th unit vector. Let $\varepsilon_n = 1$ if $x_n > 0$

and -1 else. Consider $a = e^{-\alpha n} \varepsilon_n e_n$. Then $\|a\| \leq 1$ and hence

$$e^{-\alpha n} |x_n| = |\phi(a)| \leq \|\phi\| \|a\| \leq \|\phi\|.$$

Taking the supremum over $n \in \mathbb{N}$ yields the assertion. For the converse we assume

$$\phi_{(x_n)} = \sum_n x_n a_n$$

and $\sup_n e^{\alpha n} |x_n| \leq 1$. Note that this sum is convergent because only finitely many terms are non zero. Thus we get

$$|\phi_{(x_n)}| = \left| \sum_n x_n a_n \right| = \left| \sum_n e^{-\alpha n} x_n e^{\alpha n} a_n \right|$$

$$\leq \left(\sup_n e^{-\alpha n} |x_n|\right) \sum_n e^{\alpha n} |a_n| = \left(\sup_n e^{-\alpha n} |x_n|\right) \|(a_n)\|.$$

Thus we have characterized exactly the continuous functionals of norm ≤ 1 . This is enough. ■