Practice problems for the final

Let $1 \leq p < \infty$ and f a measurable bounded function such that $f \in L_p$ show that for all $p < q < \infty$ we have $f \in L_q$ and

$$\lim_{q\to\infty} \|f\|_q \;=\; \|f\|_\infty \;.$$

Solution: Let $p < q < \infty$ and $c = \|f\|_\infty$. Note that $|f(x)| \leq c$ holds a.e.

$$\int |f(x)|^q dm \leq \int |f(x)|^p |f(x)|^{q-p} dm \;\leq\; c^{q-p} \int |f(x)|^p dm \;.$$

Thus $f \in L_q(\mathbb{R})$. For the second part, we consider a natural number $m > c$. Let $n \in \mathbb{N}$ and define the simple function

$$h_l \;=\; \sum_{k=0}^{nm} \frac{k}{n}\, 1_{\frac{k}{n}<|f|\leq \frac{k+1}{n}}$$

and

$$h^u \;=\; \sum_{k=0}^{nm} \frac{k+1}{n}\, 1_{\frac{k}{n}<|f|\leq \frac{k+1}{n}} \;.$$

Note that $h_l \leq |f| \leq h^u$ and $f \in L_q$ implies that with Chebychev that $m(\frac{k}{n} < |f| \leq \frac{k+1}{n}) < \infty$. By a previous hw problem we get for $h \in \{h_l, h^u\}$ and $a_k = k/n$ or $k + 1/n$ that

$$\lim_{p\to\infty} \|h\|_p \;=\; \lim_k (\sum_k a_k^p m(\frac{k}{n} < |f| \leq \frac{k+1}{n}))^{\frac{1}{p}} \;=\; \sup_{k:m(\frac{k}{n}<|f|\leq\frac{k+1}{n})\neq 0} a_k \;.$$

Let k_c be such that $k_c < nc \leq k_c + 1$. Thus we get

$$\lim_p \|h^u\|_p \;=\; k_c + 1/n$$

and

$$\lim_p \|h^u\|_p \;=\; k_c$$

This yields

$$\limsup_p \|f\|_p \;\leq\; \lim_p \|h^u\|_p \;\leq\; k_c + 1/n \;\leq\; c + \frac{1}{n}$$

and

$$\liminf_p \|f\|_p \;\geq\; \lim_p \|h_l\|_p \;\geq\; k_c/n \;\geq\; c - \frac{1}{n} \;.$$

Letting $n \to \infty$ we deduce the assertion. ■

Show directly that for $1 \leq p < \infty$ we have

$$\ell_p^* \;=\; \ell_{p'}$$

and that $\ell_\infty^* \neq \ell_1$.

Solution: Since we proved Holder's inequality for arbitrary measure space, we know that

$$|\sum_n a_n b_n| \;\le\; (\sum_n |a_n|^p)^{1/p} (\sum_n |b_n|^q)^{1/q}$$

whenever $1/p + 1/q = 1$, i.e. $q = p'$. Thus the mapping $u : \ell_{p'} \to \ell_p^*$ defined by $u((b_n))((a_n)) \;=\; \sum_n a_n b_n$ is satisfies

$$\|u((b_n))\|_{\ell_p^*} \;\le\; (\sum_n |b_n|^q)^{1/q} \,.$$

We will now show that u is surjective. Indeed, let $\phi : \ell_p \to \mathbb{R}$ be a linear continuous map such that

$$|\phi(a_n)| \;\le\; (\sum_n |a_n|^p)^{1/p}$$

We define

$$b_n \;=\; \phi(e_n)$$

where e_n is the n-unit vector. Let $m \in \mathbb{N}$. From the equality consideration for the Holder inequality, we deduce

$$(\sum_{n\le m} |b_n|^q)^{1/m} \;\le\; \sup_{\sum_{n\le m} |a_n|^p \le 1} \|\sum_{n\le m} a_n b_n\| \;\le\; \|\phi\| \,.$$

Taking the sup over m we get

$$(\sum_n |b_n|^q)^{1/q} \;\le\; \|\phi\| \,.$$

Since $1 \le p < \infty$, we know that simple functions are dense. However, simple function here correspond to finite sequences. By the unique extension principle we deduce that the sequence (b_n) defined above satisfies $u(b_n) = \phi$ and

$$(\sum_n |b_n|^q)^{1/q} \;\le\; \|u((b_n))\|_{\ell_p^*} \,.$$

I will assume some knowledge in logic for proving $\ell_\infty^* \ne \ell_1$. (The argument breaks down because simple function=finite sequences are no longer dense). Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Then we define

$$\phi((a_n)) \;=\; \lim_{n,\mathcal{U}} a_n$$

One can show that for every compact set and every ultrafilter $\mathcal{U}$ the limit with respect to $\mathcal{U}$ exists. Here we may consider $(a_n) \subset [-c, c]$ and the ask $A_{-}\{n \in \mathbb{N} :$

$-c \le a_n \le 0\} \in \mathcal{U}$ or $A_-^c \in \mathcal{U}$. The we split [-c,0] and (0,c] in two intervals and continue. It then easily follows that ϕ is well-define and satisfies

$$|\phi((a_n)| \ \le \ \sup_n |a_n| \ .$$

However, ϕ does not come from an element in ℓ_1. Indeed, for every $(b_n) \in \ell_1$ and every infinite set A we have

$$c_k = u((b_n))(1_{A\cap[k,\infty]}) \ = \ \sum_{n\in A, n\ge k} b_k$$

and $\sum_k |b_k|$ finite implies $\lim_k c_k \ = \ 0$. However, let $A \in \mathcal{U}$ be an infinite subset. Then

$$\phi(1_{A\cap[k,\infty]}) = 1$$

holds for all k. ■

Problem 7a) and problem 7b) on page 104.

7a) If f is a monotone increasing function, then

$$\lim_{s\uparrow t} f(s) \ = \ \sup_{s<t} f(s)$$

and

$$\lim_{s\downarrow t} f(s) \ = \ \inf_{s>t} f(s) \ .$$

Thus these limits exist. A similar argument applies for monotone deceasing function. Thus for a function of bounded variation g we deduce the result by writing $g = f_1 - f_2$ with f_i increasing.

Consider again f monotone increasing on $[a,b]$. Let A be the set of continuity points. For fixed $n \in \mathbb{N}$ we consider

$$A_n \ = \ \{t \in (a,b) : \lim_{s\uparrow t} f(s) + \frac{1}{n} \ \le \ \lim_{s\downarrow t} f(s)\}$$

Let $t_1, ..., t_m$ be m distinct elements. We may assume $a \le t_1 < t_2 < \cdots t_m \le b$. We choose points $a < s_1 < t_1 < s_2 < t_2 < s_3 < \cdots < t_m < s_m < b$. Then

$$f(b)-f(a) \ = \ f(b)-f(s_m)+f(s_m)-f(s_{m-1})+\cdots+f(s_2)-f(s_1)+f(s_1)-f(a) \ \ge \ m\frac{1}{n} \ .$$

Thus A_n has at most $n(f(b)-f(a))$ many elements. Since $\bigcup_n A_n$ is the collection of all discontinuity points in (a,b) we are done. ■

7b) Let $(r_n) \subset [0,1]$ be an enumeration of the rational points and (a_n) be a positive numbers sequence such that $\sum_n a_n \ = \ 1$. We define

$$f \ = \ \sum_n a_n 1_{[r_n,1]}$$

The f is obviously monotone and has jumps at all points r_n, i.e. $\lim_{t\to r_n, t<r_n} f(t) + a_n = \lim_{t\to r_n} f(t)$. Now, let t be an irrational point. let $\varepsilon > 0$ and n_0 such that $\sum\limits_{n>n_0} a_n < \varepsilon$. Let $\delta = \min_{j=1,\dots,n_0} |t - r_j|$. For every $|t-s| < \delta$ we have

$$|f(t) - f(s)| \;\le\; \sum_{n>n_0} |a_n| < \varepsilon \; .$$

That's it. ∎

Problem 10a) and problem 10b) on page 104.

10a) $g(x) = x^2 \cos(x^{-2})$. Let $n \in \mathbb{N}$ and define s_{n-j} such that $1/s_{n-j}^2 = \pi/2 + \pi j$ for $j = 0, ..., n$. Then we have

$$\begin{aligned} \sum_{j=0}^{n-1} |g(s_j) - g(s_{j+1})| &= \sum_{j=0}^{n-1} |s_{n-j}^2 + s_{n-j-1}^2| \\ &= \sum_{j=1}^{n} |s_j^2 + s_{j-1}^2| \\ &\ge \sum_{j=1}^{n} 1/(\pi/2 + \pi j) \;\ge\; \frac{1}{\pi} \sum_{j=1}^{n} 1/j \; . \end{aligned}$$

Since $\sum_j 1/j = \infty$ we deduce the assertion.

For b) and $g(x) = x^2 \sin(1/x)$ we note that it suffices to show that g' is in L_1. Except for 0 we have $g'(x) = 2x\sin(1/x) + x^2\cos(1/x)(-x^{-2}) = 2x\sin(1/x) - \cos(1/x)$. Thus g' is almost everywhere bounded and thus in L_1. Hence g is of bounded variation.

Problem 16) p111)-If time permits I will explain this problem Friday in class. No I didn't-but here is the solution.

a) Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Then we have

$$f(a) + \int_a^x f' dm \;\le\; f(x)$$

for every $x \in [a, b]$. Thus we may define $g(x) = f(a) + \int_0^x f' dm$ and $h(x) = f(x) - g(x)$. By the fundamental theorem (Lebesque differentiation theorem) we have $h'(x) = f'(x) - g'(x) = 0$ almost everywhere. Moreover, let $y < x$ then

$$h(x) - h(y) \;=\; f(y) - f(x) - \int_y^x f' dm \ge 0$$

be the differentiation theorem for monotone functions. Thus h is monotone and $h' = 0$ a.e.-i.e. h' is singular.

In b) and c) we prove the following

LEMMA 0.1. *Let f be a monotone function. f is singular if and only if for every $a \leq x \leq b$ and $\varepsilon > 0$ and $\delta > 0$ there exists non-overlapping intervals $[y_j, y_j + d_j]$ such that*

$$\sum_j d_j < \varepsilon \quad \textit{and} \quad f(x) - f(a) < \sum_j (f(y_j + d_j) - f(y_j)) + \delta \, .$$

PROOF. "$\Rightarrow$": Let f be singular. Let $\varepsilon > 0$ and $E = \{x : f'(x) = 0\} \cap [a+\varepsilon, b-\varepsilon]$. We may find an open subset 0 of (a, b) such that $E \subset O$ and $m(O) < m(E) + \varepsilon$. For every $x \in E$ and $\gamma > 0$ we may find $0 < h < \gamma$ such that

$$f(x + h) - f(x) < \gamma h$$

By the Vitali coverging lemma we obtain non overlapping intervals $([x_k, x_k+h_k])_{k=1,...,m}$ such that

$$\sum_{k=1}^{m} h_k > m(E \cap [a + \varepsilon, b - \varepsilon]) - \varepsilon \ \geq \ b - a - 3\varepsilon \, .$$

Without loss of generality we may assume $a + \varepsilon < x_1 < x_1 + h_1 < x_2 < x_2 + h_2 < ... < x_m < x_m + h_m \leq b$. We define $y_0 = a$ and $d_1 = x_1 - y_1$, $y_j = x_j + h_j$ and $d_j = x_{j+1} - y_j$. Finally $d_m = b - y_m$. Then

$$b - a \ = \ \sum_{k=1}^{m} d_k + \sum_{j=0}^{m} d_j \ \geq \ (b - a) - 3\varepsilon + \sum_{j=0}^{m} d_j \, .$$

This yields $\sum_j d_j < 3\varepsilon$. On the other hand

$$\begin{aligned} f(b) - f(a) &= \sum_k (f(x_k + h_k) - f(x_k)) + \sum_j f(y_j + d_j) - f(y_j)) \\ &\leq \sum_k \gamma h_k + \sum_j f(y_j + d_j) - f(y_j)) \\ &\leq \gamma(b - a) + \sum_j f(y_j + d_j) - f(y_j)) \, . \end{aligned}$$

This is exactly what we want to prove for $x = b$. However, since f' also holds on $[a, x]$ we are done.

"$\Leftarrow$": Let $f = g + h$ such that h is singular and g is absolutely continuous and $g(a) = f(a)$. Let $a \leq x \leq b$ we want to show $g(x) = g(a)$. Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\sum_k h_k < \delta \Rightarrow \sum_j (g(x_k + h_k) - g(x_k)) < \varepsilon \, .$$

(g is absolutely continuous). By assumption we find non-overlapping interval $[y_j, y_j + d_j]$ such that $\sum_j d_j < \delta$ and

$$f(x) - f(a) < \sum_j [f(y_j + d_j) - f(y_j)] + \varepsilon \,.$$

Let x_k, h_k such that $\bigcup_k [x_k, x_k + h_k] \cup \bigcup_j [y_j, y_j + h_j] = [a, b]$ and the two unions only overlap in the endpoints. Then we get

$$\begin{aligned} f(x) - f(a) &< \varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] + \sum_j [g(y_j + d_j) - g(y_j)] \\ & 2\varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] \\ & \le 2\varepsilon + \sum_j [h(y_j + d_j) - h(y_j)] + \sum_k [h(x_k + h_k) - h(x_k)] \\ & = \ 2\varepsilon + h(x) - h(a) \ = \ 2\varepsilon + h(x) \,. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we get $f(x) \ = \ h(x) + f(a)$. Thus f is singular. ∎

Proposition 0.2. *Let (f_n) positive singular monotone functions such that $f(x) = \sum_n f_n$ converges point-wise. Then f is singular.*

Proof. Without loss of generatily we may assume $f_k(a) = 0$ for all $k \in \mathbb{N}$. Let $a \le x \le b$, $\varepsilon > 0$, $\delta > 0$. Let n_0 be such that $\sum\limits_{k=1}^{n_0} f_k(x) > f(x) - \delta$. Obviously, $(\sum_{k=1}^{n_0} f_k)'(x) = \sum_{k=1}^{n_0} f_k'(x) \ = \ 0$ holds a.e. Thus we find non-overlapping intervals $([y_j, y_j + d_j])$ such that $F_{n_0} = \sum\limits_{k=1}^{n_0} f_k$ satisfies $\sum_j d_j < \delta$ and

$$F_{n_0}(x) - \varepsilon < \sum_j [F_{n_0}(y_j + d_j) - F_{n_0}(y_j)] \,.$$

This implies by monotonicity of the f_n's and by point-wise convergent that

$$\begin{aligned} f(x) - 2\varepsilon = F_{n_0}(x) &< \sum_j [F_{n_0}(y_j + d_j) - F_{n_0}(y_j)] \\ &= \sum_j \sum_{n=1}^{n_0} [f_n(y_j + d_j) - f(y_j)] \\ &= \sum_j \sum_{n=1}^{\infty} [f_n(y_j + d_j) - f(y_j)] \\ &= \sum_j [f(y_j + d_j) - f(y_j)] \,. \end{aligned}$$

Thus f is also singular-i.e. a function which creates everything out of nothing. ∎

e) Finally consider (r_n) an enumeration of the rationals and (a_n) strictly positive such that $\sum_n a_n < \infty$. Then

$$f = \sum_n a_n 1_{[r_n,1]}$$

is singular by our previous proposition. Moreover, f is strictly increasing because between two points $x < y$ we find $x < r_n < y$ and hence $f(y) - f(x) > a_n$. ■

Let $\mu << \nu$ be finite probability measures. Let f_1 and f_2 be Radon Nikodym derivatives such that

$$\mu(E) = \int\limits_E f_1 d\nu \quad \text{and} \quad \mu(E) = \int\limits_E f_2 d\nu$$

What can you say about f_1 and f_2. In which sense is the Radon-Nikodym derivative unique.

Solution: Consider $E_n = \{\omega \in \Omega : f_1(\omega) > f_2(\omega) + \frac{1}{n}\}$. Then

$$0 = \int_E (f_1 - f_2) d\nu \geq \frac{1}{n} \nu(E) \geq 0 \, .$$

Thus $\nu(E_n) = 0$. This implies $f_1 = f_2$ holds ν almost everywhere. Hence the Radon-Nikodym derivative is uniquely determined up to set of measure 0 for ν. ■

Let $\Omega = \{1, ..., n\}$ and ν the counting measure $\nu(A) = |A|$. Let μ be an arbitrary measure calculate the Radon-Nikodym derivative.

Solution: We consider the positive numbers

$$\mu_i = \mu(\{i\}) \, .$$

Define $f(i) = \mu_i$. Then

$$\mu(E) = \sum_{i \in E} \mu_i = \int_E f d\nu$$

holds for every set. ■