Midterm -Real Analysis

(1) Let $\mathcal{R}$ be an algebra and $\mu : \mathcal{R} \to [0, \infty]$ be an additive function with outer measure μ^*.

15 P) Write down the definition of a measurable set in the sense of Caratheodory.

A set A is measurable if

$$\mu * (B) = \mu * (A \cap B) + \mu^*(A^c \cap B)$$

holds for all B. The set of all such sets is called Σ_μ.

15 P) State the extension theorem which allows to construct a σ additive measure.

a) μ^* is a σ-additive measure on Σ_μ, and Σ_μ is a σ-algebra which contains $\mathcal{R}$. If in addition μ is σ-additive on $\mathcal{R}$, then $\mu^*|_{\mathcal{R}} = \mu|_{\mathcal{R}}$.

(2) (30P) State Fatou's Lemma and give an example of strict inequality. (Hint: In the "counterexample" the limit function 0 is useful and easy to obtain as a suitable for function on $\mathbb{R}$).

Fatou's Lemma: Let (f_n) be positive integrable functions. Then

$$\int \liminf_n f_n d\mu \ \le \ \liminf_n \int f_n d\mu \, .$$

The inequality is strict for

$$f_n \ = \ 1_{(n,n+1]}$$

Then $\liminf_n f_n = 0$ and

$$\liminf_n \int f_n \ = \ 1 \, .$$

(3) (40P) Let f, f_n be integrable functions on σ-finite measure space (assumed just to be on the safe side) such that f_n converges to f a.e. Show that

$$\lim_n \int |f_n| d\mu \ = \ \int |f| d\mu \quad \text{if and only if} \quad \lim_n \int |f_n - f| d\mu \ = \ 0 \, .$$

For the implication $\Leftrightarrow$ we observe that

$$|\int |f_n| - \int |f|| \ \le \ \int |f_n - f| \, .$$

Hence $\lim_n \int |f_n - f|$ show that $\lim_n \int |f_n| = \int |f|$.

For the converse, we assume that $\lim_n \int |f_n| = \int |f|$. Then

$$\int |f| = \int \liminf_n (-|f_n - f| + |f_n|) \ \le \ \liminf_n \int (-|f_n - f| + |f_n|)$$

$$
\begin{aligned}
&= \liminf_n \int (-|f_n - f| + |f_n|) \\
&= \liminf_n \int -|f_n - f| + \int |f| \, .
\end{aligned}
$$

Thus

$$
\limsup_n \int |f_n - f| \; = \; 0 \, .
$$