Math 540-Real Analysis- Homework 4

Due date: October 1-**Submission in pairs**

(1) Let (Ω, Σ, μ) be a finite measure space. Let $\mathcal{M}$ be the set of all measurable functions f on Ω such that

$$\lim_n \mu\{w : |f(\omega)| \geq n\} = 0 .$$

(a) Let $(f_n) \subset \mathcal{M}$ be sequence such that

$$d(f_n, f_{n+1}) \leq 4^{-n} .$$

Show that there exists a $f \in \mathcal{M}$ such that

$$\lim_n d(f_n, f) = 0 .$$

(b) We say that $f \sim g$ if $f = g$ holds a.e. Then $\sim$ is an equivalence relation. On $\mathcal{M}/\sim$, i.e. a set of representatives of equivalence classes we define

$$d(x, y) = d(f, g)$$

whenever $f \in x$, $g \in y$. Show that $(\mathcal{M}/\sim, d)$ is a complete metric space.

(2) (a) Let (X, d) be a complete metric space and O_n open dense. Show that $\bigcap_n O_n$ is not meager.

(b) Let $n \in \mathbb{N}$. Show that there is an open dense set $O_n \subset [0, 1]$ such that $\mu(O_n) < \frac{1}{n}$. Conclude that there is a set of measure 0 which is not meager.