Math 540-Real Analysis- Homework 4

**Due date:** October 8-**Submission in pairs**

(1) (Convex functions)

(a) Let $O \subset \mathbb{R}$ open and convex and $f : O \to \mathbb{R}$ be a differentiable function such that $f'$ is monotone increasing. Let $x < y$, $0 < \lambda < 1$ and $x_\lambda = \lambda x + (1-\lambda)y$. Use the mean value theorem to show that

$$\frac{f(x_\lambda) - f(x)}{x_\lambda - x} \;\leq\; \frac{f(y) - f(x_\lambda)}{y - x_\lambda}$$

and conclude that

$$f(\lambda x + (1-\lambda)y) \;\leq\; \lambda f(x) + (1-\lambda) f(y)\,.$$

If this holds for all $x, y, \lambda$ the function is called *convex*.

(b) Let $f$ be a convex function defined on an open neighborhood of 0 and $f(0) = 0$. Show that

$$g(x) \;=\; \frac{f(x)}{x}$$

is monotone increasing. Hint: Consider first the case $0 < x < y$ and choose a suitable $\alpha$, repeat the argument for $y < x < 0$ and then show compare $g(x_1) \leq g(x_2)$ whenever $x_1 < 0 < x_2$. The function is a priori not defined for $x = 0$.

(c) Let $f$ be a convex function defined on an open convex set $O$ and $x_0 \in O$. Show that

$$f'^{+}(x_0) \;=\; \lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and

$$f'^{-}(x_0) \;=\; \lim_{h \downarrow 0} \frac{f(x_0 - h) - f(x_0)}{-h}$$

exists (they are not necessarily the same!). Let $f'^{-}(x_0) \leq \alpha \leq f'^{+}(x_0)$. Show that

$$h(x) \;=\; f(x_0) + \alpha(x - x_0)$$

satisfies $h(x) \;\leq\; f(x)$. Hint: You may assume $x_0 = 0 = f(x_0)$ and look up.

(2) Let $0 < p < 1$.

(a) Show that $(a+b)^p \;\leq\; a^p + b^p$ holds for $a, b > 0$. (Hint: It suffices to do this for $a = 1$ and compare derivatives.)

(b) Recall $\mathcal{L}^p = \{f : f\text{measruable, a.e. finite}, \int |f|^p < \infty\}$. Show that $f, g \in \mathcal{L}^p$ implies that $f + g \in \mathcal{L}^p$. And show that

$$d_p(f, g) = \int |f - g|^p$$

satisfies the triangle inequality.

(3) No 24(page 96). Assume that $(f_n)$ converges to $f$ in measure and $|f_n|, |f| \leq g \in \mathcal{L}^1$. Then

$$\lim_n \int |f_n - f| d\mu = 0 .$$

(Hint: Use subsequences and the DCT.)