

Homework 8-Real Analysis

Due Date: November 5

- (1) Let (X, d) be complete metric space and (Ω, Σ, μ) a finite measure space.

Let

$$d_0(f, g) = \inf\{\varepsilon > 0 : \mu(\{\omega \in \Omega : d(f(\omega), g(\omega)) > \varepsilon\}) < \varepsilon\}.$$

- (a) Let $x_0 \in X$ and $f_0(\omega) = x_0$ the constant function, and

$$L_0(\Omega, \Sigma, \mu; X) = \{[f] : \int f \text{ measr. } d(f, f_0) < \infty\}.$$

Here $f \equiv g$ if $f = g$ holds a.e. Show that L_0 is complete

- (b) Show that every function $f \in L_0$ can be approximated by a simple function, i.e. a function $f : \Omega \rightarrow X$ so that $f(\Omega)$ has only finitely many values.

- (2) Let $K = \{z \in \mathbb{C} : |z_1| + |z_2| \leq 1\}$ and $z = (z_1, z_2)$. Find the point $k \in K$ such that

$$\|k - z\| = \inf_{h \in K} \|h - z\|.$$

- (3) Show that

$$2^{-n} \sum_{\varepsilon_k = \omega 1} \left\| \sum_k \varepsilon_k x_k \right\|_H^2 = \sum_k \|x_k\|_H^2$$

holds for all x_1, \dots, x_n in a Hilbert space. Why is this called generalized parallelogram inequality.

- (4) Let $H = L_2[-1, 1]$ equipped with the measure $\mu(A) = \frac{m(A)}{2}$, the normalized Lebesgue measure. Apply the Gram-Schmidt orthonormalization procedure to $F = \{f_0, f_1, f_2, f_3\}$ with

$$f_0(x) = 1 \quad , \quad f_1(x) = x \quad , \quad f_2(x) = x^3 \quad , \quad f_3(x) = x^4$$

Homework 9-Real Analysis

Due Date: November 12

- (1) Let $\Omega = \mathbb{N}$ and ν, μ given by

$$\nu(\{k\}) = \begin{cases} \frac{1}{k} & \text{if } n = 2k \\ 0 & \text{else} \end{cases}$$

and

$$\mu(\{k\}) = \begin{cases} \frac{1}{k} & \text{if } n = 3k \\ 0 & \text{else} \end{cases}$$

Find the Radon-Nikodym densities (sometimes called Radon-Nikodym derivatives for ν with respect to μ and for μ with respect to ν). Describe the exceptional sets.

- (2) Let m be the Lebesgue measure, and $\nu(A) = \int_A x dm(x)$. Let Σ_0 be the σ algebra generated by $\{[0, 1/2), [1/2, 1]\}$ and $\mu([0, 1/2)) = 1/3$, $\mu([1/2, 1]) = 2/3$. Describe the Radon-Nikodym derivative of the restriction of μ to Σ_0 .
- (3) Let H be a Hilbert space and $H_0 \subset H$ be a subspace. Show that every linear continuous map $\phi_0 : H_0 \rightarrow \mathbb{R}$ has an extension $\phi : H \rightarrow \mathbb{R}$ such that $\|\phi\| = \|\phi_0\|$ and $\phi|_{H_0} = \phi_0$.
- (4) Let X be the space of differentiable functions equipped with the norm

$$\|f\| = \left(\int_{\mathbb{R}} |f|^2 dm + \int_{\mathbb{R}} |f'|^2 dm \right)^{1/2}.$$

- i) Let g_1, g_2 be two functions in L_2 . Show that

$$\phi_{g_1, g_2}(f) = \int f g_1 dm + \int f' g_2 dm$$

defines a linear function on X with

$$\|\phi_{g_1, g_2}\| \leq (\|g_1\|_2^2 + \|g_2\|_2^2)^{1/2}.$$

- ii) Show that every linear functional on X is of the form above.
- iii) Assume that g_1, g_2 are such g_2 is differentiable and h_1, h_2 are differentiable. When is it true that

$$\phi_{g_1, g_2} = \phi_{h_1, h_2} ?$$

Homework 10-Real Analysis

Due Date: November 19

In the first two problems you may want to use simple functions.

- (1) Let (Ω, Σ, μ) be a probability space and $1 \leq p_0 < \infty$. Let $f \in L_p$ for some $p > p_0$. Show that

$$\lim_{p \rightarrow p_0, p > p_0} \|f\|_p = \|f\|_{p_0}.$$

- (2) Let (Ω, Σ, μ) be probability space and f be a measurable function such that

$$\liminf_{p \rightarrow p_0, p < p_0} \|f\|_p$$

is finite. Show that $f \in L_{p_0}$ and

$$\|f\|_{p_0} = \lim_{p \rightarrow p_0} \|f\|_p.$$

- (3) Let (Ω, Σ, μ) be a probability space. Let $\Sigma_0 \subset \Sigma$ be a sub σ -algebra. Let $P : L_2(\Omega, \Sigma, \mu) \rightarrow L_2(\Omega, \Sigma_0, \mu)$ be the orthogonal projection.

- i) Show that

$$\int P(f)gd\mu = \int fgd\mu$$

holds for all simple function $g = \sum_j \alpha_j 1_{E_j} \in S(\Sigma_0)$, i.e. $E_j \in \Sigma_0$.

- ii) Show that for $1 < p \leq 2$ and $f \in L_2(\Omega, \Sigma, \mu)$ one has

$$\|P(f)\|_p \leq \|f\|_p.$$

- iii) Show that $P : L_1(\Omega, \Sigma, \mu) \rightarrow L_1(\Omega, \Sigma_0, \mu)$ extends to a linear map with

$$\|P(f)\|_1 \leq \|f\|_1.$$

Homework 11-Real Analysis

Due Date: December 3

- (1) Problem 14 page 111.
- (2) Problem 16 page 111.
- (3) Problem 19 page 111.
- (4) Let $1 < p < \infty$. Find an element $x \in \ell_p(\mathbb{N})$ such that for no $1 < q < p$ we have

$$x \in \ell_q .$$