1. Continuous functions between metric spaces

Continuous functions 'preserve' properties of metric spaces and allow to describe deformation of one metric space into another. There are three different (but equivalent) ways of defining continuity, the ε-δ-criterion, the sequence criterion and the topological criterion. Each of them is interesting in its own right.

Definition 1.1. *Let (X, d) and (Y, d') be metric spaces. A map $f : X \to Y$ is called continuous if for every $x \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$d(x, y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon \,. \tag{1.1}$$

Let us use the notation

$$B(x, \delta) \;=\; \{y \,:\, d(x, y) < \delta\} \,.$$

For a subset $A \subset X$, we also use the notation

$$f(A) \;=\; \{f(x) \,:\, x \in A\} \,.$$

Similarly, for $B \subset Y$

$$f^{-1}(B) \;=\; \{x \in X \,:\, f(x) \in B\} \,.$$

Then (1.1) means

$$f(B(x, \delta)) \;\subset B(f(x), \varepsilon) \,.$$

Or in a very non-formal way

$$\text{f maps small balls into small balls} \;.$$

Our aim is to prove a criterion for continuity in terms of so called open sets. This criterion illustrates simultaneously the role of open sets and its interaction with continuity and has a genuinely geometric flavor.

Definition 1.2. *A subset O of a metric space is called <u>open</u> if*

$$\forall x \in O \;: \exists \delta > 0 \;: B(x, \delta) \subset O \,.$$

Examples:

$$O \;=\; (-1, 1) \,,\; O \;=\; \mathbb{R} \,,\; O \;=\; (-1, 1) \times (-2, 2)$$

are open in $\mathbb{R}$, $(\mathbb{R}^2, d_2)$ respectively.

Remark 1.3. *The sets $B(x, \varepsilon)$, $x \in X$, $\varepsilon > 0$ are open.*

PROPOSITION 1.4. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. f is contiuous iff $f^{-1}(O)$ is open for all open subsets $O \subset Y$.*

PROOF. $\Rightarrow$: We assume that f is continuous and O is open. Let $x \in f^{-1}(O)$, i.e. $f(x) \in O$. Since O is open, there exists an $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset O$. By continuity, there exists a $\delta > 0$ such that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon) \subset O \, .$$

Therefore

$$B(x, \delta) \subset f^{-1}(O) \, .$$

Since $x \in f^{-1}(O)$ was arbitrary, we deduce that $f^{-1}(O)$ is open.
$\Leftarrow$: Let $x \in X$ and $\varepsilon > 0$. Let us show that

$$B(f(x), \varepsilon)$$

is a on open subset of (Y, d'). Indeed, let $y \in B(f(x), \varepsilon)$ define $\varepsilon' = \varepsilon - d'(y, f(x))$. Let $z \in Y$ such that

$$d(z, y) < \varepsilon'$$

then

$$d(f(x), z) \;\leq\; d(f(x), y) + d(y, z) \;<\; d(f(x), y) + \varepsilon - d'(y, f(x)) \;=\; \varepsilon \, .$$

Thus

$$B(y, \varepsilon - d'(f(x), y)) \;\subset B(f(x), \varepsilon) \, .$$

By the assumption, we see that $f^{-1}(B(f(x), \varepsilon))$ is an open set. Since $x \in f^{-1}(B(f(x), \varepsilon))$, we can find a $\delta > 0$ such that

$$B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \, .$$

Hence, for all $\tilde{x}$ with $d(x, \tilde{x}) < \delta$, we have

$$d'(f(x), f(\tilde{x})) < \varepsilon \, .$$

The assertion is proved. ■

Examples:

(1) Let (X, d) be a metric space and $x_0 \in X$ be a point , then $f(x) = d(x, x_0)$ is continuous. Indeed, the triangle inequality implies

$$d(d(x, x_0), d(d(y, 0)) \;=\; |d(x, x_0) - d(y, x_0)| \;\leq\; d(x, y)$$

This easily implies the assertion.

(2) On $\mathbb{R}^n$ with the standard euclidean metric $d = d_2$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = d(x, 0)x$ is continuous.

(3) (Exercise) The function $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\cos(x_1), \sin(x_2), \cos(x_1))$ is continuous.

Definition 1.5. *Let (X, d), (Y, d') be a metric space. The space $C(X, Y)$ is the set of all continuous functions from X to Y. Let $x_0 \in X$ be a point. Then*

$$C_b(X, Y) \;=\; \{f : X \to Y \;:\; f \text{ is continuous and } \sup_{x\in X} d'(f(x), f(x_0)) < \infty\}$$

is the subset of bounded continuous functions.

Proposition 1.6. *Let (X, d), (Y, d') be metric spaces and $x_0 \in X$. Then $C_b(X, Y)$ equipped with*

$$d(f, g) \;=\; \sup_{x\in X} d'(f(x), g(x))$$

is a metric space.

Problem: Show that d is not well-defined on $C(\mathbb{R}, \mathbb{R})$.

Proof: $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all $x \in X$. This means $f = g$. Let us show that d is well-defined. Indeed, if $f, g \in C_b(X, Y)$. Then

$$\begin{aligned}\sup_x d'(f(x), g(x)) &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + d'(g(x_0), g(x)) \\ &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + \sup_x d(g(x_0), g(x))\end{aligned}$$

is finite. Let h be a third function and $x \in X$. Than

$$d'(f(x), g(x)) \;\le\; d'(f(x), h(x)) + d(h(x), g(x)) \;\le\; d(f, h) + d(h, g)\;.$$

Taking the supremum yields the assertion. ■

Proposition 1.7. *Let (X, d) be a metric space. Then $C(X, \mathbb{R})$ is closed under (pointwise-) sums, products and multiplication with real numbers. ($C(X, \mathbb{R})$ is an algebra over $\mathbb{R}$).*

Remark 1.8. *Let $X = \mathbb{N}$ and $d(x, y) = 1$ of $x \neq y$ and $d(x, y) = 0$ for $x = y$. (This is called the discrete metric). Then $C(X, \mathbb{R})$ is an infinite dimensional vector space.*

Proof of 1.7. Let $f, g \in C(X, \mathbb{R})$ be continuous and $x \in X$. Consider $x' \in X$. Then

$$fg(x) - fg(y) \;=\; f(x)g(x) - f(y)g(y) \;=\; (f(x) - f(y))g(x) + f(y)(g(x) - g(y))$$

$$= (f(x) - f(y))g(x) + f(x)(g(x) - g(y)) + (f(y) - f(x))(g(x) - g(y))\ .$$

Let $\varepsilon > 0$ and $\tilde{\varepsilon} = \min\{\varepsilon, 1\}$. We may choose $\delta_1 > 0$ such that

$$d(f(x), f(y))(1 + |g(x)|) < \frac{\tilde{\varepsilon}}{3}$$

holds for all $d(x, y) < \delta_1$. Similarly, we may choose $\delta_2 > 0$ such that

$$d(g(x), g(y))(1 + |f(x)|) < \frac{\tilde{\varepsilon}}{3}\ .$$

Let $\delta = \min(\delta_1, \delta_2)$ and $d(x, y) < \delta$. Then we deduce that

$$d(fg(x), fg(y)) = |fg(x) - fg(y)| < \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}^2}{9} < \tilde{\varepsilon} \ \leq\ \varepsilon\ .$$

Thus fg is again continuous. The other assertions are easier. ■

Corollary 1.9. *The polynomials on $\mathbb{R}$ are continuous.*

Lemma 1.10. *Let $1 \leq p \leq \infty$ and $x, y \in \mathbb{R}^n$, then*

$$\frac{1}{n^{\frac{1}{p}}}\, d_p(x, y) \ \leq\ d_\infty(x, y) \ \leq\ d_p(x, y)\ .$$

Proof. The last inequality is obvious. For the first one, we consider $x, y \in \mathbb{R}^n$ and $1 \leq p < \infty$, then by estimating every element in the sum against the maximum

$$d_p(x, y)^p \ = \ \sum_{i=1}^{n} |x_i - y_i|^p \ \leq\ n \max\{|x_i - y_i|^p\}\ .$$

Taking the p-th root, we deduce the assertion. ■

Corollary 1.11. *Let $1 \leq p, q \leq \infty$, then the identity map $id : (\mathbb{R}^n, d_p) \to (\mathbb{R}^n, d_q)$ is continuous.*

Proof. We have for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$

$$B_{d_p}(x, \frac{\varepsilon}{n}) \subset B_{d_q}(x, \varepsilon)\ .$$

This easily implies the assertion. ■

Corollary 1.12. *The metrics d_p define the same open sets on $\mathbb{R}^n$.*

DEFINITION 1.13. *Let (X, d) be a metric space. We say that a sequence (x_n) converges to x_0 if for all $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ we have*

$$d(x_n, x_0) < \varepsilon .$$

In this case we write

$$\lim_n x_n \; = \; x$$

or more explicitly

$$d - \lim_n x_n \; = \; x .$$

A sequence (x_n) is convergent, if there exists $x \in X$ with $\lim_n x_n = x$.

Examples: $d_2 - \lim_n \frac{1}{n} = 0$, $dd_3 - \lim_n 3^n = 0$. (What axioms of the natural numbers are involved?).

PROPOSITION 1.14. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. Then f is continuous if for every convergent sequence (x_n) in X*

$$\lim_n f(x_n) \; = \; f(\lim_n x_n) .$$

Proof: $\Rightarrow$: Let $x = \lim_n x_n$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that

$$d(y, x) < \delta \Rightarrow d'(f(y), f(x)) < \varepsilon .$$

Let $n_0 \in \mathbb{N}$ be such that

$$d(x_n, x) < \delta$$

for all $n > n_0$, then

$$d'(f(x_n), f(x)) < \varepsilon$$

for all $n > n_0$. Hence

$$\lim_n f(x_n) \; = \; f(x) .$$

$\Leftarrow$ Let $x \in X$ and assume in the contrary that

$$\exists \varepsilon > 0 \, \forall \delta > 0 \exists y \; : d(y, x) < \delta \text{ and } d'(f(x), f(y)) \geq \varepsilon .$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence (x_k) such that

$$d(x_k, x) \; < \; \frac{1}{k} \quad \text{and} \quad d'(f(x_k), f(x)) \geq \varepsilon' .$$

and thus

$$\lim_k x_k \; = \; x .$$

By assumption, we have

$$\lim_k f(x_k) \; = \; f(x) .$$

Hence, there exists a k_0 such that for all $k > k_0$

$$d(f(x_k), f(x)) < \varepsilon \ .$$

a contradiction. ■

2. Complete metric spaces and completion

Complete metric space are crucial in understanding existence of solutions to many equations. Complete spaces are also important in understanding spaces of integrable functions. We will review basic properties here and show the existence of a completion.

We will say that a sequence in a metric space is a Cauchy sequence of for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$.

Definition 2.1. *A metric space (X, d) is called complete, if every Cauchy sequence converges.*

Proposition 2.2. *The space $(\mathbb{R}^2, d_1)$ is complete.*

Proof: Let x_n be a Cauchy sequence in $(\mathbb{R}^2, d_1)$. Then $x_n = (x_n(1), x_n(2))$ is a sequence of pairs.

Claim: The sequences $(x_n(1))_{n\in\mathbb{N}}$ and $(x_n(2))_{n\in\mathbb{N}}$ are Cauchy sequences.

Indeed, let $\varepsilon > 0$, then there exists an n_0 such that

$$d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$. In particular, we have

$$|x_n(1) - x_m(1)| \;\le\; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \;\le\; d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$ and

$$|x_n(2) - x_m(2)| \;\le\; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \;\le\; d_1(x_n, x_m) < \varepsilon \,.$$

Therefore, $(x_n(1))$ and $(x_n(2))$ are Cauchy.

Since $\mathbb{R}$ is complete, we can find $x(1)$ and $x(2)$ such that

$$\lim_n x_n(1) \;=\; x(1) \quad \text{and} \quad \lim_n x_n(2) \;=\; x(2) \,.$$

Claim: $\lim_n x_n \;=\; (x(1), x(2))$.

Indeed, Let $\varepsilon > 0$ and choose n_1 such that

$$|x_n(1) - x(1)| < \frac{\varepsilon}{2}$$

for all $n > n_1$. Choose n_2 such that

$$|x_n(2) - x(2)| < \frac{\varepsilon}{2}$$

for all $n > n_2$. Set $n_0 = \max\{n_1, n_2\}$, then for every $n > n_0$, we have

$$d_1(x_n, (x(1), x(2)) \;=\; |x_n(1) - x(1)| + |x_n(2) - x(2)| \;<\; \varepsilon$$

Thus

$$\lim_n x_n \;=\; x$$

and the assertion is proved. ∎

Examples:

(1) Let $X = \mathbb{R} \setminus \{0\}$ and $d(x,y) = |x - y|$, them (X, d) is not complete. The sequences $(\frac{1}{n})$ is Cauchy and does not converge.

(2) Let p be a prime number. On the set of integers, we define

$$dd_p(z, w) \;=\; p^{-n} \,,$$

where $n = \max\{n \;:\; p^n \text{ divides } (z-w)\;\}$. This satisfies the triangle inequality. The sequence (x_n) given by $x_n = p + p^2 + \cdots + p^n$ is a non convergent Cauchy sequence.

THEOREM 2.3. *Let $n \in \mathbb{N}$. The space $(\mathbb{R}^n, d_2)$ is a complete metric space.*

PROOF. Similar as in Proposition 2.2 using the following Lemma . ∎

LEMMA 2.4. *Let $x, y \in \mathbb{R}^n$, then*

$$d_2(x,y) \;\leq\; \sum_{i=1}^{n} |x_i - y_i| \,.$$

PROOF. We proof this by induction on $n \in \mathbb{N}$. The case $n = 1$ is obvious. Assume the assertion is true for n and let $x, y \in \mathbb{R}^{n+1}$. We define the element $z = (x_1, ..., x_n, y_{n+1})$, then we deduce from the triangle inequality

$$\begin{aligned} d_2(x,y) \;&\leq\; d_2(x,z) + d_2(z,y) \\ &=\; \left(\sum_{i=1}^{n+1} |x_i - z_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n+1} |z_i - y_i|^2\right)^{\frac{1}{2}} \\ &=\; |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} \,. \end{aligned}$$

To apply the induction hypothesis, we define $\tilde{x} = (x_1, ..., x_n)$ and $\tilde{y} = (y_1, .., y_n)$. Then the induction hypothesis yields

$$\left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}} \;=\; d_2(\tilde{x}, \tilde{y}) \;\leq\; \sum_{i=1}^{n} |x_i - y_i| \, .$$

Hence,

$$\begin{aligned} d_2(x, y) \;&\leq\; |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}} \\ &\leq\; |x_n - y_n| + \sum_{i=1}^{n} |x_i - y_i| \\ &=\; \sum_{i=1}^{n+1} |x_i - y_i| \, . \end{aligned}$$

The assertion is proved. ■

Definition 2.5. *A subset $C \subset X$ is called closed if $X \setminus C$ is open.*

Proposition 2.6. *Let C be closed subset of a complete metric space (X, d), then $(C, d|_{C \times C})$ is complete.*

Proof. Let $(x_n) \subset C$ be Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$x \;=\; \lim_{n} x_n \, .$$

We have to show $x \in C$. Assume $x \notin C$. Then there exists a $\delta > 0$ such that $B(x, \delta) \subset X \setminus C$. By definition of the limit there exists n_0 such that $d(x_n, x) < \delta$ for all $n > n_0$. Set $n = n_0 + 1$. Then $d(x_n, x) < \delta$ implies $x_n \in X \setminus C$ and $x_n \in C$ by definition. This contradiction finished the proof. ■

Theorem 2.7. *Let (Y, d') be complete metric space. Let $h \in C(X, Y)$ and*

$$C_h(X, Y) \;=\; \{ f \in C(X, Y) \; : \; \sup_{x \in X} d'(f(x), h(x)) < \infty \}$$

Then $C_g(X, Y)$ is complete with respect to

$$d(f, g) \;=\; \sup_{x \in X} d'(f(x), g(x)) \, .$$

Proof. Let $(f_n) \subset C_h(X,Y)$ be Cauchy sequence. This means that for every $\varepsilon > 0$ there exists an n_0 such that

$$\sup_{x\in X} d'(f_n(x), f_m(x)) < \frac{\varepsilon}{2} \,. \tag{2.1}$$

In particular, for fixed $x \in X$, $f_n(x)$ is Cauchy. Therefore $f(x) := \lim_m f_m(x)$ is a well-defined element in Y. We fix $n > n_0$ and consider $m \geq n_0$ such that

$$d'(f_m(x), f(x)) \ \leq \ \frac{\varepsilon}{3} \,.$$

This implies

$$d'(f_n(x), f(x)) \ \leq \ d'(f_n(x), f_m(x)) + d'(f_m(x), f(x)) \ \leq \ \frac{5}{6}\varepsilon$$

for all $x \in X$. In particular,

$$\sup_{n\geq n_0} \sup_{x\in X} d'(f_n(x), f(x)) \ \leq \ \frac{5}{6}\varepsilon \,. \tag{2.2}$$

Let us show that f is continuous. Let $z \in X$ and $\varepsilon > 0$. Choose n_0 according to (2.1). Choose $n = n_0 + 1$. Let $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(f_n(x), f_n(y)) < \varepsilon$. Then, we have

$$d'(f(x), f(y)) \ \leq \ d'(f(x), f_n(x)) + d'(f_n(x), f_n(y)) + d'(f_n(y), f(y)) < 3\varepsilon \,.$$

Since $\varepsilon > 0$ is arbitrary, we see that f is continuous. Moreover, (2.2) implies that f_n converges to f. Finally, (2.2) for $\varepsilon = 1$ implies that

$$\sup_x d(f(x), h(x)) \ \leq \ \sup_x d(f(x), f_n(x)) + \sup_x d(f_n(x), h(x)) < \infty$$

implies that $f \in C_h(X,Y)$. ■

Definition 2.8. *Let (X,d) be a metric space and $C \subset X$. $O \subset X$ is called sense if for ever $x \in C$ and $\varepsilon > 0$ $B(x,\varepsilon) \cap O \neq \emptyset$.*

Definition 2.9. *Let $O \subset X$ be a subset. Then*

$$\bar{O} \ = \ \cap_{O\subset C, C\, closed} C$$

is called the closure.

Lemma 2.10. *O is dense in $\bar{O}$ and $\bar{O}$ is closed.*

PROOF. Let $x \in \bar{O}$. Assume $B(x,\varepsilon) \cap O = \emptyset$. Then $C = X \setminus B(x,\varepsilon)$ contains O. Thus

$$\bar{O} \subset C \,.$$

This implies that $x \notin \bar{O}$, a contradiction. Now, we show that $\bar{O}$ is closed. Indeed, let $y \notin \bar{O}$. Then there has to be a closed set C such that $O \subset C$ but $y \notin C$. This means $y \in X \setminus C$ which is open. Hence there exists $\delta > 0$ such that

$$B(y,\delta) \subset X \setminus C$$

By definition every element $z \in B(y,\delta)$ does not belong to $\bar{O}$. This means $B(y,\delta) \subset X \setminus \bar{O}$. ■

THEOREM 2.11. *Let (X,d) be a non-empty metric space. For every $x \in X$ we define*

$$f_x(y) \;=\; d(x,y) \,.$$

Let $x_0 \in X$. The map $f : X \to C_{f_{x_0}}(X,\mathbb{R})$ satisfies the following properties.

i) $d(f(x), d(f(y)) \;=\; d(x,y)$,
(1) *The closure $C = \overline{f(X)}$ is complete,*
(2) *$f(X)$ is dense in the closure $C = \overline{f(X)}$.*

PROOF. Let $x, y \in X$ and $z \in X$. Then the 'converse triangle' ineqquality implies

$$|f_x(z) - f_y(z)| \;=\; |d(x,z) - d(y,z)| \;\leq\; d(x,y) \,.$$

Moreover,

$$|f_x(z) - f_x(y)| \;=\; |d(x,z) - d(x,y)| \;\leq\; d(z,y) \,.$$

Therefore $f_x \in C_{f_{x_0}}(X,\mathbb{R})$ for every $x \in X$ and

$$d(f_x, f_y) \;\leq\; d(x,y) \,.$$

However,

$$d(f_x, f_y) \;\geq\; |f_x(x) - f_y(x)| \;=\; |0 - d(y,x)| \;=\; d(y,x) \,.$$

This shows i). According to Proposition 2.6 and Theorem 2.7, we see that C is complete. According to Lemma 2.10, we deduce that $f(X)$ is dense in C. ■

Project: On $C([0,1])$ we define

$$d_1(f,g) \;=\; \int |f(s) - g(s)| ds \,.$$

Show that $(C([0,1]), d_1)$ is not complete.

Project: In the literature you can find another description of the completion of a metric space. Find it and describe it.

3. Unique extension of densely defined uniformly continuous functions

In this section we will show that the completion C constructed in Theorem 2.11 is unique (in some sense). This is based on a simple observation-the unique extension. This principle is very often used in analysis.

DEFINITION 3.1. *Let (X,d), (Y,d') be metric spaces. A function $f : X \to Y$ is called uniform continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$d(x,y) < \delta \quad \Rightarrow \quad d'(f(x),f(y)) < \varepsilon \, .$$

PROPOSITION 3.2. *Let $O \subset C$ be a dense set and $f : O \to Y$ be uniformly continuous function with values in a complete metric space. Then there exists a unique continuous function $\tilde{f} : O \to Y$ such that $\tilde{f}(x) = f(x)$ for all $x \in O$.*

PROOF. Let $x \in X$. Since $B(x, \frac{1}{n}) \cap O$ is not empty, we may find $(x_n) \subset O$ such that $\lim_n x_n = x$. We try to define

$$\tilde{f}(x) \;=\; \lim_n f(x_n) \, .$$

Let us show that this is well-defined. So we consider another Cauchy sequence (x'_n) such that $\lim_n x'_n = x$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$d'(f(x',y)) < \varepsilon$$

holds for $d(x',y) < \delta$. We may find n_0 such that

$$d(x_n,x) < \frac{\delta}{2}$$

and

$$d(x'_n,x) < \frac{\delta}{2}$$

holds for all $n,n' > n_0$. Thus

$$d'(f(x'_n),f(x_n)) < \varepsilon \, .$$

This argument also shows that $(f(x_n))$ is Cauchy and hence $\tilde{f}(x)$ is well-defined. If $x \in O$, we may choose for (x_n) the constant sequence $x_n = x$ and hence $\tilde{f}(x) = f(x)$. Now, we want to show that $\tilde{f}$ is uniformly continuous. Indeed, let $\varepsilon > 0$, then there exists $\delta > 0$ such that $d(x',y') < \delta$ implies

$$d(f(x'),f(y')) < \frac{\varepsilon}{2} \, .$$

Given $x, y \in C$ with $d(x,y) < \delta$, we may find (x_n) converging to x and (y_n) converging to y such that

$$d(x_n, x) < \frac{\delta - d(x,y)}{2} \ .$$

Thus for all $n \in \mathbb{N}$ we have

$$d(x_n, y_n) \ \le \ d(x,y) + d(x_n, x) + d(y_n, y) < \delta \ .$$

This implies

$$d(f(x), f(y)) \ = \ \lim_n d(f(x_n), f(y_n)) < \frac{\varepsilon}{2} \ .$$

This shows that $\tilde{f}$ is uniformly continuous. If g is another continuous function such that $g(x) = f(x)$ holds for elements $x \in O$, then we may choose a Cauchy sequence (x_n) converging to x and get

$$g(x) = \lim_n g(x_n) \ = \ \lim_n f(x_n) \ = \ f(x) \ . \qquad \blacksquare$$

Example If $f : (0,1] \to \mathbb{R}$ is uniformly continuous, then f is bounded (why). $f(x) = 1/x$ is not uniformly continuous.

THEOREM 3.3. *The completion of a metric space is unique. More precisely, let C be the set constructed in Theorem 2.11. Let C' be a complete metric space and $\iota' : X \to C'$ be uniformly continuous with uniformly continuous inverse ${\iota'}^{-1} : \iota(X) \to X$ such that $\iota'(X)$ is dense. Then there is a bijective, bicontinuous map $u : C \to C'$ such that $u(\iota(x)) = \iota'(x)$.*

PROOF. The map $\iota'\iota^{-1} : \iota(X) \to C'$ is uniformly continuous and hence admits a unique continuous extension $u : C \to C'$. Also $\iota{\iota'}^{-1} : \iota'(X) \to C$ admits a unique extension $v : C' \to C$. Note that $vu : C \to C$ is an extension of the map $vu(\iota(x)) = \iota(x)$. Thus there is only one extension, namely the identity. This show $vu = id$. Similarly $uv = id$. Thus $v = u^{-1}$ and u is bijective and bi-continuous. $\blacksquare$

Project: Find the completion of $(\mathbb{Z}, d_3)$.

4. Closed and Compact Sets

Let (X, d) be a metric space. We will say that a subset $A \subset X$ is *closed* if $X \setminus A$ is open.

Proposition 4.1. *Let (X, d) be a complete metric space and $C \subset X$ a subset. C is closed iff every Cauchy sequence in C converges to an element in C.*

Proof: Let us assume C is closed and that (x_n) is a Cauchy sequence with elements in C. Let $x = \lim_n x_n$ be te limit and assume $x \notin C$. Since $X \setminus C$ is open

$$B(x, \varepsilon) \subset X \setminus C$$

for some $\varepsilon > 0$. Then there exists an n_0 such that $d(x_n, x) < \varepsilon$ for $n > n_0$. In particular,

$$x_{n_0+1} \in B(x, \varepsilon)$$

and thus $x_{n_0+1} \notin C$, a contradiction.
Now, we assume that every Cauchy sequence with values in C converges to an element in C. If $X \setminus C$ is not open, then there exists an $x \notin C$ and no $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset X \setminus C \, .$$

I.e. for every $n \in \mathbb{N}$, we can find $x_n \in C$ such that

$$d(x, x_n) < \frac{1}{n} \, .$$

Hence, $\lim x_n = x \in C$ but $x \notin C$, contradiction. ■

The most important notion in this class is the notion of compact sets. We will say that a subset $C \subset X$ is *compact* if For every collection (O_i) of open sets such that

$$C \subset \bigcup_i O_i \; = \; \{x \in X \mid \exists_{i \in I} x \in O_i\}$$

There exists $n \in \mathbb{N}$ and $i_1, ..., i_n$ such that

$$C \subset O_{i_1} \cup \cdots \cup O_{i_n} \, .$$

In other words

Every open cover of C has a finite subcover .

Definition 4.2. *Let $X \subset \bigcup O_i$ be an open cover. Then we say that (V_j) is an open subcover if*

$$X \subset \bigcup_j V_j$$

all the V_j are open and for every j there exists an i such that

$$V_j \subset O_i \, .$$

It is impossible to explain the importance of 'compactness' right away. But we can say that there would be no discipline 'Analysis' without compactness. The most clarifying idea is contained in the following example.

Proposition 4.3. *The set $[0,1] \subset \mathbb{R}$ is compact.*

Proposition 4.4. *Let $B \subset X$ be closed set and $C \subset X$ be a compact set, then*

$$B \cap C$$

is compact

Proof: Let $B \cap C \subset \bigcup O_i$ be an open cover. then

$$C \subset (X \setminus B) \cup \bigcup_i O_i$$

is an open cover for C, hence we can find a finite subcover

$$C \subset (X \setminus B) \cup O_{i_1} \cup \cdots \cup O_{i_n} \, .$$

Thus

$$B \cap C \ \subset O_{i_1} \cup \cdots \cup O_{i_n}$$

is a finite subcover. ■

Lemma 4.5. *Let (X,d) be a metric space and $D \subset X$ be a countable dense set in X, then for every subset $C \subset X$ and every open cover*

$$C \subset \bigcup_i O_i$$

we can find a countable subcover of balls.

Proof: Let us enumerate D as $D = \{d_n \,|\, n \in \mathbb{N}\}$. Let $x \in C$ and find $i \in I$ and $\varepsilon > 0$ such that

$$x \in B(x, \varepsilon) \subset O_i \,.$$

Let $k > \frac{2}{\varepsilon}$. By density, we can find an $n \in \mathbb{N}$ such that

$$d(x, d_n) < \frac{1}{2k} \,.$$

Then

$$x \in B(d_n, \frac{1}{2k}) \subset B(x, \frac{1}{k}) \subset B(x, \varepsilon) \subset O_i \,.$$

Let us define

$$M \;=\; \{(n,k) \mid \exists_{i \in I} B(d_n, \frac{1}{2k}) \subset O_i\} \,.$$

Then $M \subset \mathbb{N}^2$ is countable and hence there exists a map $\phi : \mathbb{N} \to M$ which is surjective (=onto). Hence for $V_m \;=\; B(d_{\phi_1(m)}, \frac{1}{2\phi_2(m)})$, ϕ_1, ϕ_2 the 2 components of ϕ we have

$$C \subset \bigcup_m V_m$$

and (V_m) is a countable subcover of balls of the original cover (O_i). ■

THEOREM 4.6. *Let (X, d) be a metric space. Let $C \subset X$ be a subset. Then the following are equivalent*

i) a) *Every Cauchy sequence of elements in C converges to a limit in C.*
   b) *For every $\varepsilon > 0$ there exists points $x_1, ..., x_n \in X$ such that*

$$C \subset B(x_1, \varepsilon) \cup \cdots \cup B(x, \varepsilon) \,.$$

ii) *Every sequence in C has a convergent subsequence.*
iii) *C is compact.*

Proof: *i)* $\Rightarrow$ *ii)*. Let (x_n) be a sequence. Inductively, we will construct infinite subset $A_1 \supset A_2 \supset A_3 \cdots$ and y_1, y_2, y_3,... in X such that

$$\forall_{l \in A_j} \;:\; d(x_l, y_j) < 2^{-j-1} \,.$$

Put $A_0 = \mathbb{N}$. Let us assume $A_1 \supset A_2 \supset \cdots A_n$ and $y_1, ..., y_n$ have been constructed. We put $\varepsilon = 2^{-n-2}$ and apply condition *i)b)* to find $z_1, ..., z_m$ such that

$$C \;\subset\; B(z_1, \varepsilon) \cup \cdots \cup B(z_m, \varepsilon) \,.$$

We claim that there must be a $1 \le k \le m$ such that

$$A_n(k) \;=\; \{l \in A_n \,|\, x_n \in B(z_k, \varepsilon)\}$$

has infinitely many elements. Indeed, we have

$$A_n(1) \cup \cdots \cup A_n(m) \;=\; A_n \;.$$

If they were all finite, then a finite union of finite sets would have finitely many elements. However A_n is infinite. Contradiction! Thus, we can find a k with $A_n(k)$ infinite and put $A_{n+1} = A_n(k)$ and $y_{n+1} = z_k$. So the inductive procedure is finished. Now, we can find an increasing sequence (n_j) such that $n_j \in A_j$ and deduce

$$d(x_{n_j}, x_{n_{j+1}}) \;\le\; d(x_{n_j}, y_j) + d(y_j, x_{n_{j+1}}) < \frac{1}{2}2^{-j} + \frac{1}{2}2^{-j} \;=\; 2^{-j}$$

because $n_j \in A_j$ and $n_{j+1} \in A_{j+1} \subset A_j$. Thus (x_{n_j}) is Cauchy. Indeed, be induction, we deduce for $j < m$ that

$$\begin{aligned} d(x_{n_j}, x_{n_m}) &\le d(x_{n_j}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_{n_{j+2}}) \cdots d(x_{n_{m-1}}, x_{n_m}) \\ &\le 2^{-j} \sum_{k=0}^{m-1} 2^{-k} \;=\; 2^{1-j} \;. \end{aligned}$$

This easily implies the Cauchy sequence condition. By a) it converges to some $x \in C$. We got our convergent subsequence.
$ii) \Rightarrow iii)$: We will first show $ii) \Rightarrow i)b)$. Indeed, let $\varepsilon > 0$ and assume for all $n \in \mathbb{N}$, $y_1, ..., y_n \in C$ we may find

$$x(n, y_1, ..., y_n) \in C \setminus (B(y_1, \varepsilon) \cup \cdots B(y_n, \varepsilon)) \;.$$

Then we define $x_1 \in C$ and find $x_2 \in C \setminus B(x_1, \varepsilon)$. Then we find $x_3 \in C \setminus B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$. Thus inductively we find $x_n \in C$ such that

$$d(x_n, x_k) \ge \varepsilon$$

for all $1 \le k \le n$. It is easily seen that (x_n) has no convergent subsequence. Thus $i)b)$ is showed (with points in C). For every $\varepsilon_k = \frac{1}{k}$ we find these points $y_1^k, ..., y_{m(k)}^k \in C$ such that

$$C \subset B(y_1^k, \frac{1}{k}) \cup \cdots \cup B(y^{m(k)}, \frac{1}{k}) \;.$$

Then, we see that $D = \{y_j^k : k \in \mathbb{N}, 1 \le j \le m(k)\}$ is dense in C. Therefore, we may work with the closure $\tilde{X} = \bar{D}$ and show that C is compact in $\tilde{X}$. (It will then be automatically compact in X). By Lemma 4.5, we may assume that

$$C \subset \bigcup_k O_k$$

and O_k's open. If we can find an n such that

$$C \subset O_1 \cup \cdots \cup O_n$$

the assertion is proved. Assume that is not the case and choose for every $n \in \mathbb{N}$ an $x_n \in C \setminus O_1 \cup \cdots \cup O_n$. According to the assumption, we have a convergent subsequence, i.e. $\lim_k x_{n_k} = x \in C$. Then $x \in O_{n_0}$ for some n_0 and there exists a $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset O_{n_0} .$$

By convergence, we find a k_0 such that $d(x, x_{n_k}) < \varepsilon$ for all $k > k_0$. In particular, we find a $k > k_0$ such that $n_k > n_0$. Thus

$$x_{n_k} \in B(x, \varepsilon) \in O_{n_0} \subset O_1 \cup \cdots \cup O_{n_k} .$$

Contradicting the choice of the (x_n)'s. We are done.
iii) $\Rightarrow$ *i)b)* Let $\varepsilon > 0$ and then

$$C \subset \bigcup_{x \in C} B(x, \varepsilon) .$$

thus a finite subcover yields *b)*.
iii) $\Rightarrow$ *i)a)* Let (x_n) be a Cauchy sequence. Assume it is not converging to some element $x \in C$. This means

$$\forall x \in C \exists \varepsilon(x) > 0 \forall n_0 \exists n > n_0 \; d(x_n, x) > \varepsilon . \tag{4.1}$$

Then

$$C \subset \bigcup_{x \in C} B(x, \frac{\varepsilon(x)}{2}) .$$

Let

$$C \subset B(y_1, \frac{\varepsilon(y_1)}{2}) \cup \cdots \cup B(y_1, \frac{\varepsilon(y_1)}{2})$$

be a finite subcover (compactness). Then there exists at least one $1 \leq k \leq m$ such that

$$A_k = \{n \in \mathbb{N} \mid d(x_n, y_k) < \frac{\varepsilon(y_k)}{2}\}$$

is infinite. Fix that k and apply the Cauchy criterion to find n_0 such that

$$d(x_n, x_{n'}) < \frac{\varepsilon(y_k)}{2}$$

for all $n, n' > n_0$. By (4.1), we can find an $n > n_0$ such that

$$d(x_n, y_k) > \varepsilon(y_k) .$$

Since A_k is infinite, we can find an $n' > n_0$ in A_k thus

$$\begin{aligned} \varepsilon(y_k) &< d(x_{n'}, y_k) \leq d(x_n, x_{n'}) + d(x_{n'}, y_k) \\ &< \frac{\varepsilon(y_k)}{2} + \frac{\varepsilon(y_k)}{2} = \varepsilon(y_k) . \end{aligned}$$

A contradiction. Thus the Cauchy sequence has to converge to some point in C. ■

COROLLARY 4.7. *Every intervall $[a,b] \subset \mathbb{R}$ with $a < b \in \mathbb{R}$ is compact*

Proof: It is easy to see that $X \setminus [a,b]$ is open. Hence, by Proposition 4.1 $[a,b]$ is complete, i.e. i)a) is satisfied. Given $\varepsilon > 0$, we can find $k > \frac{1}{\varepsilon}$. For $m > k(b-a)$ we derive

$$[a,b] \subset \bigcup_{j=0}^{m} B(a + \frac{j}{k}, \varepsilon) .$$

Thus the Theorem 4.6 applies. ■

LEMMA 4.8. *Let $r > 0$ and $n \in \mathbb{N}$, the set $C_r = [-r,r]^n$ is compact.*

Proof: Let $x \notin C_r$, then there exists an index $j \in \{1,..,n\}$ such that $|x_j| > r$. Let $\varepsilon = |x_j| - r$ and $y \in \mathbb{R}^n$ such that

$$\max_{i=1,..,n} |x_i - y_i| < \varepsilon ,$$

then

$$|y_j| \;=\; |y_j - x_j + x_j| \;\geq\; |x_j| - |y_j - x_j| \;>\; |x_j| - \varepsilon \;=\; r .$$

thus $y \notin C_r$. Hence, C_r is closed and according to Proposition 2.3, we deduce that C_r is complete.
For $n = 1$ and $\varepsilon > 0$, we have seen above that for $k > \frac{1}{\varepsilon}$ and $m > \frac{2r}{k}$

$$[-r,r] \subset \bigcup_{j=0}^{m} B(-r + \frac{j}{k}, \varepsilon) .$$

Therefore

$$[-r,r]^n \subset \bigcup_{j_1,...,j_n=0,...m} B_\infty((-r + \frac{j_1}{k}, .., -r + \frac{j_n}{k}), \varepsilon) .$$

Thus i)a) and i)b) are satisfies and the Theorem 4.6 implies the assertion (The separable dense subset is $\mathbb{Q}^n$.) ■

THEOREM 4.9. *Let $C \subset \mathbb{R}^n$ be a subset. The following are equivalent*

1) *C is compact.*
2) *C is closed and there exists an r such that*

$$C \subset B(0,R) .$$

(That is C is bounded.)

Proof: 2) $\Rightarrow$ 1) Let

$$C \subset B(0,R) \subset [-R,R]^n$$

be a closed set. Since $[-R,R]^n$ is compact, we deduce from Proposition 4.4 that C is compact as well.

1) $\Rightarrow$ 2) Let C subset $\mathbb{R}^n$ be a compact set. According to Theorem 4.6 i)b), we find

$$C \subset B(x_1,1) \cup \cdots \cup B(x_m,1)$$

thus for $r = \max_{i=1,..,m}(d(x_i,0)+1)$ we have

$$C \subset B(0,r) \, .$$

Moreover, by Theorem 4.6 i)a) and Proposition 4.1, we deduce that C is closed. ■

We will now discuss one of the most important applications.

THEOREM 4.10. *Let (X,d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. The there exists $x_0 \in X$ such that*

$$f(x_0) \;=\; \sup\{f(x) : x \in X\} \, .$$

PROOF. Let us first assume

$$A = \{f(x) : x \in X\}$$

is bounded and $s = \sup A$. For every $n \in \mathbb{N}$, we know that $s - \frac{1}{n}$ is no upper bound. Hence there $x_n \in X$ such that

$$s \geq f(x_n) > s - \frac{1}{n} \, .$$

Let (n_k) be such that $\lim_k x_{n_k} = x \in X$. Then we deduce from continuity that

$$f(x) \;=\; \lim_k f(x_{n_k}) \;\geq\; \lim s - \frac{1}{n_k} \;=\; s \, .$$

By definition of s we find $f(x) = s$. Now, we show that A is bounded. Indeed, if note we find $x_n \in X$ such that $f(x_n) \geq n$. Again we find a convergent subsequence (x_{n_k}). Since $f(x_{n_k})$ is convergent it is bounded. We assume (f_{n_k}) is bounded above by $m \in nz$. Choosing $k \geq m+1$ we get

$$m \geq f(x_{n_k}) \geq n_k > n_m \geq m \, .$$

This contradiction shows that A is bounded and hence the first argument applies. ■

5. C(K)

For a metric space X, we denote by $C(X)$ the space of continuous functions with values in the real numbers.

LEMMA 5.1. *Let (X,d) be a metric space and $f,g \in C(X)$ and $t \in \mathbb{R}$, then*

i) *$f+tg$ defined by $f+tg(x) = f(x)+tg(x)$, $x \in X$, is in $C(X)$.*
ii) *fg defined by $fg(x) = f(x)g(x)$, $x \in X$, is in $C(X)$.*
iii) *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function, then $h \circ f$ defined by $h \circ f(x) = h(f(x))$ is continuous.*
iv) *Let (f_n) be a sequence of continuous functions such that for every eps > 0 there exists n_0 such that for all $n,m > n_0$*

$$\sup_{x\in X} |f_n(x) - f_m(x)| \;\le\; \varepsilon \;,$$

Then there is a continuous function $f : X \to \mathbb{R}$ such that

$$f(x) \;=\; \lim_n f_n(x) \;.$$

Proof: *iii)* Let us assume that (f_n) is a sequence as above. Clearly for all $x \in X$, we see that

$$f(x) \;=\; \lim_n f_n(x)$$

exists. We have to show that f is continuous. For let $x \in X$ and $\varepsilon > 0$. Let n_0 be chosen according such that for $n,m > n_0$

$$\sup_{x\in X} |f_n(x) - f_m(x)| \;\le\; \frac{\varepsilon}{3} \;.$$

In particular, for all $y \in X$ and for $n = n_0 + 1$, we deduce

$$|f(y) - f_n(y)| \;=\; \lim_m |f_m(y) - f_n(y)| \;\le\; \frac{\varepsilon}{3} \;. \tag{5.1}$$

Since, f_n is continuous (at x), we can find $\delta > 0$ such that $d(x,y) < \delta$ implies

$$|f_n(x) - f_n(y)| \;<\; \frac{\varepsilon}{3} \;.$$

Thus we get for those y

$$\begin{aligned} |f(x) - f(y)| \;&\le\; |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(x)| \\ &<\; \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \;=\; \varepsilon \;. \end{aligned}$$

Hence f is continuous and the assertion is proved. ■

THEOREM 5.2. *Let K be a compact metric space and $f : K \to \mathbb{R}$ be a continuous function. Then there exists an $x_0 \in K$ such that*

$$f(x_0) \;=\; \sup\{f(x) \mid x \in K\}\,.$$

Proof: Note that in the proof of the main Theorem the existence of a countable dense subset has only been used to prove $ii) \Rightarrow iii)$. Thus, we still have that every sequence in a compact space has a convergent subsequence. If $\sup\{f(x) \,:\, x \in K\} = \infty$, we can find a sequence (x_n) such that

$$f(x_n) > n$$

for all $n \in \mathbb{N}$. Let us consider this case first. Let (x_{n_k}) be a convergent subsequence with

$$x \;=\; \lim_k x_{n_k} \in K\,.$$

Then we can find an $\varepsilon > 0$ such that

$$d(x,y) < \varepsilon \quad \Rightarrow \quad |f(x) - f(y)| \;\le\; 1$$

Then there exists an k_0 such that $d(x_{n_k}, x) < \varepsilon$ for $k > k_0$. Let k_1 be such that $k_1 > |f(x)| + 2$, then we deduce for some $k_2 > \max\{k_1, k_2\}$

$$|f(x)| + 1 \;\;\le\;\; k_2 \;\le\; f(x_{k_2}) \;\le\; f(x) + |f(x) - f(x_{k_2})| \;<\; |f(x)| + 1\,.$$

a contradiction. Thus $\sup\{f(x) \,:\, x \in K\} < \infty$. For every $\varepsilon > 0$, there exists an $x(\varepsilon) \in K$ such that

$$f(x(\varepsilon)) > \sup\{f(x) \,:\, x \in K\}\,.$$

Call the supremum *sup*. We get a sequence (x_n) such that

$$f(x_n) \le sup \;\le\; f(x_n) + \frac{1}{n}$$

Let (x_{n_k}) be a convergent subsequence with

$$x \;=\; \lim_k x_{n_k} \in K\,.$$

By continuity of f, we deduce

$$f(x) \;=\; \lim_k f(x_{n_k}) \;=\; sup\,.$$

The assertion is proved. ■

Corollary 5.3. *Let K be a compact set and $f : K \to \mathbb{R}$ be a continuous function, then*

$$\sup\{f(x) : x \in K\}$$

is finite.

Corollary 5.4. *The space $C(K)$ equipped with*

$$d(f,g) \; = \; \sup_{x \in K} |f(x) - g(x)|$$

is a complete metric space.

Proof: Let us fist observe that for $f, g \in C(K)$ the map $|f - g|$ is continuous and thus

$$d(f,g)$$

is a real number. Clearly, d is symmetric and $f = g$, i.e. $f(x) = g(x)$ for all $x \in K$ iff $d(f,g) = 0$. The triangle inequality is obvious. Indeed, let $f, g, h \in C(K)$. Then

$$\begin{aligned} \sup_{x \in K} |f(x) - g(x)| &\leq \sup_{x \in K} |f(x) - h(x) + h(x) - g(x)| \\ &\leq \sup_{x \in K} (|f(x) - h(x)| + |h(x) - g(x)|) \\ &\leq \sup_{x \in K} |f(x) - h(x)| + \sup_{x \in K} |h(x) - g(x)| \\ &= d(f,h) + d(h,g) \; . \end{aligned}$$

Given a Cauchy sequence (f_n), we apply Lemma 5.1 to obtain a continuous limit function f. According to (5.1), we see that

$$\lim_{n \to \infty} d(f, f_n) \; = \; 0 \; .$$

Hence, f_n converges to f. (Details: Exercise.) ∎

Motivated by this result, we will say that a sequence of functions (f_n) converges uniformly to f (on X) if

$$\forall \varepsilon > 0 \exists n_0 \forall n > n_0 \forall x \in X |f_n(x) - f(x)| \; < \; \varepsilon \; .$$

This opposed to the pointwise convergence

$$\forall x \in X \forall \varepsilon > 0 \exists n_0 \forall n > n_0 |f_n(x) - f(x)| \; < \; \varepsilon \; .$$

Example: The functions $f_n(x) = x^n$ converge pointwise to $f(x) = 0$ on $[0,1)$ However, the following remarkable result allows us to show that under suitable circumstances the weaker pointwise convergence implies uniform convergence.

THEOREM 5.5. *Let K be a compact space. (f_n) a sequence of continuous functions on K converging pointwise to the* continuous *function f on K such that for all $x \in K$ the sequence $f_n(x)$ is increasing. Then (f_n) converges uniformly to f.*

Proof: Let $\varepsilon > 0$. Then, we can find for every $x \in K$ an $n(x)$ such that

$$f(x) < f_n(x) + \frac{\varepsilon}{3} \,.$$

Since f, $f_{n(x)}$ are continuous, we can find $\delta(x) > 0$ such that

$$\delta(x,y) < \delta(x) \qquad \Rightarrow \qquad (|f(x) - f(y)| < \frac{\varepsilon}{3} \;\; \& \;\; |f_{n(x)}(x) - f_{n(x)}(y)| < \frac{\varepsilon}{3} \,.$$

Then we have an open over

$$X \subset \bigcup_{x \in K} B(x, \frac{\delta(x)}{2})$$

by compactness can find a finite subcover

$$X \subset B(x_1, \delta(x_1)) \cup \cdots \cup B(x_m, \delta(x_m)) \,.$$

Let $n(\varepsilon) = \max\{n(x_1), ..., n(x_m)\}$. Let $y \in K$ and find an $1 \leq i \leq m$ such that $d(x_i, y) < \delta(x_i)$. Then, we have for $x = x_i$

$$\begin{aligned} f(y) &\leq \; f(x) + \frac{\varepsilon}{3} \; < \; f_{n(x)} + \frac{2\varepsilon}{3} \\ &\leq \; f_n(x) + \frac{2\varepsilon}{3} \; \leq \; f_n(y) + \varepsilon \,. \end{aligned}$$

By montonicity, we deduce for all $m > n$

$$f(y) < f_m(y) + \varepsilon \; \leq \; f(y) + \varepsilon \,.$$

The assertion is proved. ■

Let us state a further important application of compactness.

THEOREM 5.6. *Let $K \subset (X, d)$ be a compact subset and $f : K \to (Y, d)$ be a continuous function. Then for every $\varepsilon > 0$ there exists an $\delta > 0$ such that for all $x, y \in K$*

$$d(x,y) < \delta \quad \Rightarrow \quad d(f(x), f(y)) \; < \; \varepsilon \,.$$

Proof: For every $x \in K$, we can find $\delta(x)$ such that

$$d(x,y) < \delta \quad \Rightarrow \quad d(f(x), f(y)) \; < \; \frac{\varepsilon}{2} \,.$$

Then the open

$$X \subset \bigcup_{x \in X} B(x, \frac{\delta(x)}{2})$$

has a finite subcover

$$X \subset B(x_1, \frac{\delta(x_1)}{2}) \cup \cdots \cup B(x_m, \frac{\delta(x_m)}{2})\,.$$

Let $\delta = \min_{i=1,..,m}\{\frac{\delta(x_i)}{2}$ and consider $x, y \in K$ such that $d(x,y) < \delta$. Then there exists $1 \le i \le m$ such that $d(x, x_i) < \frac{\delta(x_i)}{2}$. Hence,

$$d(y, x_i) \;\le\; d(y,x) + d(x, x_i) \;<\; \delta(x_i)$$

and thus

$$d(f(x), f(y)) \;\;\le\;\; d(f(x), f(x_i)) + d(f(x_i), f(y)) \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon\,.$$

The assertion is proved. ■

Definition 5.7. *A function satisfies the assertion of the previous theorem is called uniformly continuous.*

If time remains, we will show:

Theorem 5.8. *The closure of the polynomials are dense in* $C([a,b])$.

Proof: $ii)$: Let $x \in X$ and $0 < \varepsilon < 1$, then there exists a $\varepsilon_f > 0$ such that

$$d(x,y) < \varepsilon_f \quad \Rightarrow \quad d(f(x), f(y)) \; < \; \frac{\varepsilon}{3(1+|g(x)|)} \; < \; 1\,,$$

and $\varepsilon_g > 0$ such that

$$d(x,y) < \varepsilon_g \quad \Rightarrow \quad d(g(x), g(y)) \; < \; \frac{\varepsilon}{2(1+|g(x)|)} \; < \; 1\,.$$

Let $y \in X$ such that $d(y,x) < \min\{\varepsilon_f, \varepsilon_g, 1\} = \delta$. Then we deduce form $\varepsilon < 1$

$$\begin{aligned} |fg(x) - fg(y)| &\leq |f(x)||g(x)-g(y)| + |f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||g(x)-g(y)| + |f(x)-f(y)||g(x)| + |f(x)-f(y)||g(x)-g(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon^2}{9} \; < \; \varepsilon\,. \end{aligned}$$

6. Some remarks on the ternary Cantor function

The complement of the Cantor set in $[0,1]$ is given by

$$O = \bigcup_{n=0}^{\infty} \bigcup_{a \in \{0,2\}^n} O_a$$

where for $n=0$, $a=\emptyset$

$$O_a = (\frac{1}{3}, \frac{2}{3})$$

and for $n \geq 0$, $a=(a_1,...,a_n)$ we have

$$O_a = (\sum_{i=1}^{n} a_i 3^{-i} + 3^{-(n+1)}, \sum_{i=1}^{n} a_i 3^{-i} + 23^{-(n+1)}) .$$

On such a set O_a we define

$$f(x) = \sum_{i=1}^{n} \frac{a_i}{2} 2^{-i} + 2^{-(n+1)}$$

For $n=0$ we use the value $\frac{1}{2}$.

LEMMA 6.1. *f is uniformly continuous.*

PROOF. Let us consider x_1 and x_2 in O such that $|x_1 - x_2| < 3^{-n}$. We may write

$$x_1 = \sum_{i=1}^{\infty} a_i 3^{-i}$$

and

$$x_2 = \sum_{i=1}^{\infty} b_i 3^{-i}$$

such that there exists a smallest integer k and m with $a_k = 1$, $b_m = 1$. Let j be the smallest integer such that $a_j \neq b_j$.

case 1) $j < \min(k,m)$: W.l.o.g. we may assume $a_j = 2$ and $b_j = 0$. Then

$$3^{-n} \geq x_1 - x_2 \geq 23^{-j} - 3^{-j} = 3^{-j} .$$

This yields $j \geq n$ and

$$|f(x_1) - f(x_2)| \leq \sum_{i \leq j} |a_i - b_i| 2^{-i} \leq 22^{-j} \leq 42^{-n} .$$

case 2) $j \geq \min(k,m)$. If $k=m$, then $f(x_1) = f(x_2)$ are we are fine. Let us assume $k > m$. Then $a_m \in \{0,2\}$ and $b_m = 1$. Thus $j = m$.

case 2a) $a_m = 2$: Let us fix the smallest $j > k$ such that $a_j \neq 0$. We get

$$f(x_1) - f(x_2) \;=\; \sum_{i=m+1}^{k-1} \frac{a_i}{2} 2^{-i} + 2^{-m} \,.$$

Moreover,

$$3^{-n} x_1 - x_2 \geq 3^{-k} + a_j 3^{-j} - (\sum_{i=k+1}^{\infty} b_i 3^{-i}) \geq a_j 3^{-j} \,.$$

Thus $j \geq n$ and hence

$$|f(x_1) - f(x_2)| \;\leq\; \sum_{i \geq j} \frac{a_i}{2} 2^{-i} + 2^{-m} \;\leq\; 42^{-j} \;\leq\; 42^{-n} \,.$$

case 2b) $a_m = 0$: similar. ∎

Integrable functions-Outline of statements

7. The space of integrable functions

In the following (Ω, Σ, μ) is a σ-finite measure space.

DEFINITION 7.1. *$f \in S(\mu)$ if $f : \Omega \to \mathbb{R}$ is a measurable function such that $f(\Omega)$ is a finite set and*

$$\mu(f \neq 0) < \infty .$$

Then

$$I(f) = \sum_{r \in f(\Omega)} r\mu(\{f = r\}) .$$

PROPOSITION 7.2. *$I : S(\mu) \to \mathbb{R}$ is linear. Moreover, $f \leq g$ implies $I(f) \leq I(g)$. Furthermore, we have the triangle inequality*

$$I(|f - g|) \leq I(|f - h|) + I(|h - g|) .$$

PROOF. 1. Note that $I(\lambda g) = \lambda I(g)$. Now if $f, g \in S(\mu)$, then

$$f = \sum_{i=0}^{n} x_i 1_{E_i} \text{ where } E_i = f^{-1}(\{x_i\})$$
$$g = \sum_{i=0}^{n} y_i 1_{F_i} \text{ where } F_i = f^{-1}(\{y_i\}),$$

where $x_0 = y_0 = 0$. We assume that $x_i \neq x_k (i \neq k) \Rightarrow E_i \cap E_k = \emptyset$ and $\cup E_i = \Omega$, and $y_j \neq y_l (j \neq l) \Rightarrow F_j \cap F_l = \emptyset$ and $\cup F_j = \Omega$. Consider $\{x_i + y_j : 0 \leq i \leq n, 0 \leq j \leq m\} = \{Z_0, Z_1, \ldots, Z_N\}$, then

$$\begin{aligned}
f + g &= \sum_{r=0}^{N} Z_r 1_{\cup_{x_i + y_j = z_r} E_i \cap E_j} \\
&= \sum_{r=0}^{N} Z_r \mu(\cup_{x_i + y_j = z_r} E_i \cap E_j) \\
&= \sum_i \sum_j (x_i + y_j)\mu(E_i \cap E_j) \\
&= \sum_{i=1}^{n} x_i \sum_{j=0}^{m} \mu(E_i \cap F_j) + \sum_{j=1}^{m} y_j \sum_{i=0}^{n} \mu(E_i \cap F_j) \\
&= \sum_i x_i \mu E_i + \sum_j y_j \mu F_j \\
&= I(f) + I(g)
\end{aligned}$$

2.

$$\begin{aligned} I(f) &= \sum_{i=0}^{n}\sum_{j=0}^{m} x_i \mu(E_i \cap F_j) \leq \sum_{i=0}^{n}\sum_{j=0}^{m} y_j \mu(E_i \cap F_j) \\ &= \sum_{j=1}^{n} y_j \sum_{i=0}^{n} \mu(E_i \cap F_j) = \sum_{j=1}^{n} y_j F_j = I(g) \end{aligned}$$

3.

$$I(f) - I(g) = I(f-g) \leq I(|f-g|) \Rightarrow |I(f) - I(g)| \leq I(|f-g|)$$

■

DEFINITION 7.3. *Let* $f : \Omega \to [0, \infty]$ *a positive measurable function. Then*

$$I(f) \;=\; \sup_{0 \leq h \leq f, h \in S(\mu)} I(h)\,.$$

f *is called (positive) integrable if* $I(f)$ *is finite.*

LEMMA 7.4. *1)* $f \leq g$*, then* $I(f) \leq I(g)$*. 2)* $I(f+g) \;\geq\; I(f) + I(g)$

PROOF. 2) Let $0 \leq f_n \leq f$ and $0 \leq g_n \leq g \Rightarrow 0 \leq f_n + g_n \leq f + g$. Then,

$$I(f) + I(g) = \sup I(f_n) + \sup I(g_n) \leq I(f+g).$$

■

We want to show that for integrable f, g we have $I(f+g) \leq I(f) + I(g)$.

LEMMA 7.5. $f \geq 0$ *integrable,* $0 \leq h \leq f$*,* $h \in S(\mu)$*. Then*

$$I(f) \;=\; I(f-h) + I(h)\,.$$

PROOF. Let $\epsilon > 0$ and $0 \leq \tilde{g} \leq f$, $\tilde{g} \in S(\mu)$ such that

$$I(\tilde{g}) \leq I(f) \leq I(\tilde{g}) + \epsilon.$$

Define $g = \max\{\tilde{g}, h\} \geq \tilde{g}$, then

$$I(g) \leq I(f) \leq I(\tilde{g}) + \epsilon \leq I(g) + \epsilon = I(g-h) + I(h) + \epsilon \leq I(f-h) + I(h) + \epsilon.$$

■

COROLLARY 7.6. $f \geq 0$, $I(f) \leq \infty$. *Then there exists an increasing sequence of simple functions* $\{g_n\}$ *such that* $0 \leq g_n \leq f$ *and* $\lim I(f - g_n) = 0$.

PROOF. Let $\epsilon = 2^{-n}$. Furthermore, let $0 \leq \tilde{g}_n \leq f$ and $I(f) \leq I(\tilde{g}_n) + \epsilon_n$. Define $g_n = \max\{\tilde{g}_1, \dots, \tilde{g}_n\}$, then

$$I(g_n) \leq I(f) \leq I(\tilde{g}_n) + \epsilon_n \leq I(g_n) + \epsilon_n.$$

By Lemma 9.5:

$$I(g_n) + \epsilon \geq I(f) = I(f - g_n) + I(g_n) \Rightarrow I(f - g_n) \leq \epsilon_n.$$

■

LEMMA 7.7. *$f \geq 0$ integrable. Then there exits a sequence $f_n \in S(\mu)$ such that $0 \leq f_n \leq f$, f_n is increasing and*

$$I(f - f_n) \ \leq \ 4^{-n} \, .$$

LEMMA 7.8. *(Chebychev) f integrable, $\lambda > 0$ and Ω is σ-finite. Then*

$$\lambda\mu(f \geq \lambda) \ \leq \ I(f) \, .$$

PROOF. Let $h = \lambda 1_{f \geq \lambda} \leq f$ and let $\Omega_n \subset \Omega$ an increasing sequences of subsets such that $\cup\Omega_n = \Omega$. Then

$$h_n = \lambda 1_{\{f \geq \lambda\} \cap \Omega_n} \leq I(f),$$

so

$$\mu(f \geq \lambda) = \lim \mu(\{f \geq \lambda\} \cap \Omega_n) = \lim_n \frac{1}{\lambda} I(h_n) \leq \frac{1}{\lambda} I(f).$$

■

LEMMA 7.9. *Let (f_n) and f be measurable such that*

$$I(|f_n - f|) \ \leq \ 4^{-n}$$

Then f_n converges to f almost everywhere.

PROOF. Let $E_n = \{\omega : |f - f_n| > 2^{-n}\}$, $F_n = \cup_{k \geq n} E_k$ and $F = \cap_n F_n$, then

$$\mu(F) \leq \lim_n \mu(F_n) \leq \lim_n \sum_{k \geq n} \mu(E_k) = \lim \sum_{k \geq n} 2^{-k} = \lim 2^{-n} = 0.$$

If $\omega \notin F \Rightarrow \exists n$ such that $\forall \ k \geq n$, $f(\omega) - 2^{-k} \leq f_k(\omega) \leq f(\omega) + 2^{-k}$, which implies

$$\lim_k f_k(\omega) = f(\omega).$$

■

LEMMA 7.10. *(Beppo-Levi)* $0 \leq f_n \leq f_{n+1}$ *all integrable such that* $\lim_n I(f_n) < \infty$. *If* f_n *converges to* f *alomst everywhere, then*

$$I(f) \;\leq\; \lim_n I(f_n)\;.$$

PROOF. Let $0 \leq h \leq f$, $h \in S(\mu)$ and

$$h = \sum_{i=1}^{m} r_i 1_{E_i}$$

Let $F \subset \Omega$ such that $\lim f_n(\omega) = f(\omega)\ \forall\ \omega \in F$. Given any $\epsilon > 0$, define

$$E_{i,n} = \left\{\omega \in E_i \cap F : f_n(\omega) > \frac{r_i}{1+\epsilon}\right\} \subset E_i,$$

then $E_i \cap F = \cup_n E_{i,n}$ because $\lim_n f_n(\omega) = f(\omega) \geq r_i$. We may find n such that $\mu(E_{i,n}) \geq \frac{\mu(E_i \cap F)}{1+\epsilon}$ for all $i = 1, \ldots, m$.
Then define

$$h^{\epsilon,n} = \sum_i \frac{r_i}{1+\epsilon} 1_{E_{i,n}} \leq f_n 1_F \leq f_n \Rightarrow I(h^{\epsilon,n}) \leq I(f_n) \leq \lim_n I(f_n).$$

Moreover

$$\begin{aligned} I(h) &= \sum_{i=1}^{m} r_i \mu(E_i) = \sum_{i=1}^{m} r_i \mu(E_i \cap F) \\ &\leq (1+\epsilon)^2 \sum_{i=1}^{m} \frac{r_i}{1+\epsilon} \mu(E_{i,n}) \\ &\leq (1+\epsilon)^2 I(h^{\epsilon,n}) \leq (1+\epsilon)^2 I(f_n). \end{aligned}$$

■

COROLLARY 7.11. *(Fatou) Let* $0 \leq f_n$ *be positive integrable functions. Then*

$$I(\liminf_n f_n) \;\leq\; \liminf_n I(f_n)\;.$$

PROOF. Let $g_n = \inf_{m \geq n} f_m$ is increasing. Then

$$I(\sup_n g_n) \;=\; \sup_n I(g_n) \;\leq\; \sup_n \inf_{m\geq n} I(f_m) \;=\; \liminf_n I(f_n)\;.$$

■

PROPOSITION 7.12. *f, g positive integrable. Then*

$$I(f) + I(g) \;=\; I(f+g)\;.$$

PROOF. Let $f_n \in S(\mu)$, $g_n \in S(\mu)$ increasing sequences such that $I(f-f_n) \leq 4^{-n}$ and $I(g-g_n) \leq 4^{-n}$. Then the sequence $h_n = f_n + g_n$ converges to $f+g$ almost everywhere and

$$I(f+g) \;\leq\; \lim_n I(f_n+g_n) \;=\; \lim_n I(f_n)+I(g_n) \;=\; I(f)+I(g)\;.$$

■

DEFINITION 7.13. *A measurable function $f : \Omega \to [-\infty,\infty]$ is called integrable if there exists a sequence (f_n) in $S(\mu)$ such that*

$$\lim_n I(|f-f_n|) \;=\; 0\;.$$

We denote $I(\mu)$ the space of integrable functions

PROPOSITION 7.14. *Let f be μ-integrable and (f_n), (f'_n) such that*

$$\lim_n I(|f-f_n|) \;=\; 0 \;=\; \lim_n I(|f-f'_n|)$$

Then

$$\lim_n I(|f_n-f'_n|) \;=\; 0\;.$$

In particular,

$$\int f \;=\; \lim_n I(f_n)$$

is well-defined.

PROOF.

$$I(|f_n-f'_n|) \leq I(|f_n-f|+|f'_n-f|) = I(|f_n-f|)+I(|f'_n-f|).$$

Also note

$$I(|f_n-f_m|) \leq I(|f_n-f|+|f_m-f|) = I(|f_n-f|)+I(|f_m-f|),$$

so $\lim_n I(f_n)$ exists. ■

LEMMA 7.15. *Let $f \geq 0$ be μ-integrable. Then*

$$I(f) \;=\; \int f\;.$$

PROOF. Let (f_n) be a sequence of simple functions such that $I(|f-f_n|) \leq 4^{-n}$. Then f_n converges to f almost everywhere. For fixed $n \in \mathbb{N}$ we consider $E_n = \{\omega : f_n(\omega) < 0$. Then

$$1_{E_n}|f-f_n| \;=\; 1_{E_n}|f|+1_{E_n}|f_n| \;\geq\; 1_{E_n}|f|\;.$$

Thus we have $I(|f - f_n^+|) \leq I(|f - f_n|) \leq 4^{-n}$. Then f_n^+ converges to f μ-almost everywhere and Fatou's lemma implies

$$I(f) \leq \liminf_n I(f_n^+) \leq \sup_n I(|f_n|) .$$

However,

$$\begin{aligned}|I(|f_n|) - I(|f_m|)| = |I(|f_n| - |f_m|)| &\leq I(||f_n| - |f_m||) \leq I(|f_n - f_m|) \\ &\leq I(|f_n - f|) + I(|f - f_m|) \leq 4^{-n} + 4^{-m} .\end{aligned}$$

Thus $I(|f_n|)$ is Cauchy and hence $\sup_n I(|f_n|)$ bounded. We get

$$I(h) \leq \sup_n I(|f_n|) .$$

In particular, $I(f)$ is finite. Equality follows from the preceding Proposition. ■

PROPOSITION 7.16. *1) $f, g \in I(\mu)$, $\lambda \in \mathbb{R}$. Then $\int (f + \lambda g) = \int f + \lambda \int g$.*
2) $f, g \in I(\mu)$, $f \leq g$. Then $I(f) \leq I(g)$.

PROOF. 1) There exist $I(|f_n - f|) \to 0$ and $I(|g_n - g|) \to 0$, then

$$I(|f_n + \lambda g_n - (f + \lambda g)|) \leq I(|f_n - f|) + \lambda I(|g_n - g|) \to 0.$$

Since integral is well defined, so $\int f + \lambda g = \lim_n I(f_n + \lambda g_n) = I(f) + \lambda g$.
2) Consider $h = f - g \geq 0$, then $\int h = I(h) \geq 0$. Thus

$$\int f = \int (f - g + g) = \int h + \int g \geq \int g.$$

■

PROPOSITION 7.17. *f is integrable iff f^+ and f^- are integrable. Moreover, we have*

$$\int f = \int f^+ - \int f^- .$$

PROOF. "$\Leftarrow$": $f = f^+ - f^-$.
"$\Rightarrow$": There exists f_n such that

$$\lim I(|f - f_n|) = 0 \Rightarrow \lim_n I(||f| - |f_n||) \Rightarrow 0,$$

so $|f|$ is integrable.

$$\int f = \int \frac{|f| + f}{2} + \int \frac{f - |f|}{2} = \int f^+ - \int f^-.$$

■

8. Convergence Theorems

LEMMA 8.1. *(Fatou) Let* (f_n) *be positive integrable functions. Then*

$$\int \liminf_n f_n \;\le\; \liminf_n \int f_n \,.$$

THEOREM 8.2. *(Dominated convergence theorems) Let* $f \geq 0$ *be positive integrable function. Let* (g_n) *be integrable functions such that*

$$|g_n| \le g \quad \mu \; a.e$$

for all $n \in \mathbb{N}$ *and* $g = \lim_n g_n$ *exists. Then* g *is integrable and*

$$\int g \;=\; \lim_n \int g_n \,.$$

PROPOSITION 8.3. *A bounded function is Riemann integrable iff the set of discontinuity points has measure* 0.

9. The space of integrable functions

In the following (Ω, Σ, μ) is a measure space.

Definition 9.1. *$f \in S(\mu)$ if $f : \Omega \to \mathbb{R}$ is a measurable function such that $f(\Omega)$ is a finite set and*

$$\mu(f \neq 0) < \infty \, .$$

Then

$$I(f) \;=\; \sum_{0 \neq r \in f(\Omega)} r\mu(\{f = r\}) \, .$$

Proposition 9.2. *$I : S(\mu) \to \mathbb{R}$ is linear. Moreover, $f \leq g$ implies $I(f) \leq I(g)$. Furthermore, we have the triangle inequality*

$$I(|f-g|) \leq I(|f-h|) + I(|h-g|) \, .$$

Proof. 1. Note that $I(\lambda g) = \lambda I(g)$. Now if $f, g \in S(\mu)$, then

$$f = \sum_{i=0}^{n} x_i 1_{E_i} \text{ where } E_i = f^{-1}(\{x_i\})$$

$$g = \sum_{i=0}^{n} y_i 1_{F_i} \text{ where } F_i = f^{-1}(\{y_i\}),$$

where $x_0 = y_0 = 0$. We may assume that

$$i \neq k \Rightarrow E_i \cap E_k = \emptyset \quad \cup E_i = \Omega \, .$$

Moreover, for $i \neq 0$ we know that $\mu(E_i) < \infty$. Similarly, we may assume

$$j \neq l \Rightarrow F_j \cap F_l = \emptyset \quad \cup F_j = \Omega \, .$$

Consider

$$\{x_i + y_j : 0 \leq i \leq n, 0 \leq j \leq m\} \;=\; \{Z_0, Z_1, \ldots, Z_N\} \, .$$

We assume that $Z_0 = 0$. Note that $E_0 \cap F_0 \subset (f+g)^{-1}(0)$. Then

$$\begin{aligned} I(f+g) &= \sum_{r=1}^{N} Z_r \mu(\cup_{x_i+y_j=Z_r} E_i \cap F_j) \\ &= \sum_{i,j,\min(i,j)>0} (x_i + y_j)\mu(E_i \cap F_j) \\ &= \sum_{i=1}^{n} x_i \sum_{j=0}^{m} \mu(E_i \cap F_j) + \sum_{j=1}^{m} y_j \sum_{i=0}^{n} \mu(E_i \cap F_j) \\ &= \sum_i x_i \mu(E_i) + \sum_j y_j \mu(F_j) \;=\; I(f) + I(g) \, . \end{aligned}$$

2. If $f \le g$, then $E_i \cap F_j \neq \emptyset$ implies $x_i \le y_j$. Let $G = \bigcup_{i=1}^n E_i)$. This yields

$$\begin{aligned} I(f) = \sum_{i=1}^n x_i \mu(E_i) &= \sum_{i=1}^n x_i \sum_{j=0, E_i \cap F_j \neq \emptyset}^m \mu(E_i \cap F_j) \\ &\le \sum_{i>1 E_i \cap F_j \neq \emptyset} y_j \mu(E_i \cap F_j) = \sum_{j=0}^m y_j \sum_{i=1}^m \mu(E_i \cap F_j) \\ &= \sum_{j=1}^m y_j \sum_{i=1}^n \mu(E_i \cap F_j) = \sum_{j=1}^m y_j \mu(F_j \cap G) \\ &= I(g 1_G) \, . \end{aligned}$$

However, if $\omega \in G^c$, then $f(\omega) = 0$ and hence $g(\omega) \ge 0$. This yields

$$I(g 1_{G^c}) \ \ge \ 0 \, .$$

Therefore $I(f) \le I(g1_G) \le I(g1_G) + I(g1_{G^c}) \ = \ I(g)$.
3. We note $f - g \le |f - g$ and hence

$$I(f) - I(g) \ = \ |I(f-g)| \le I(|f-g|) \, .$$

Similarly, $I(g) - I(f) \le I(|g-f|) \ = \ I(|f-g|)$. The assertion follows. ■

Definition 9.3. *Let $f : \Omega \to [0, \infty]$ a positive measurable function. Then*

$$I(f) \ = \sup_{0 \le h \le f, h \in S(\mu)} I(h) \, .$$

f is called (positive) integrable if $I(f)$ is finite.

Lemma 9.4. *1) $f \le g$, then $I(f) \le I(g)$. 2) $I(f+g) \ \ge \ I(f) + I(g)$*

Proof. 2) Let $0 \le f_n \le f$ and $0 \le g_n \le g \Rightarrow 0 \le f_n + g_n \le f + g$. Then,

$$I(f) + I(g) = \sup I(f_n) + \sup I(g_n) \le I(f+g) \, .$$ ■

We want to show that for integrable f, g we have $I(f+g) \le I(f) + I(g)$. This will be done in several steps.

Lemma 9.5. *$f \ge 0$ integrable, $0 \le h \le f$, $h \in S(\mu)$. Then*

$$I(f) \ = \ I(f-h) + I(h) \, .$$

PROOF. Let $\epsilon > 0$ and $0 \leq \tilde{g} \leq f$, $\tilde{g} \in S(\mu)$ such that

$$I(\tilde{g}) \leq I(f) \leq I(\tilde{g}) + \epsilon.$$

Define $g = \max\{\tilde{g}, h\} \geq \tilde{g}$, then

$$I(g) \leq I(f) \leq I(\tilde{g}) + \epsilon \leq I(g) + \epsilon = I(g-h) + I(h) + \epsilon \leq I(f-h) + I(h) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this concludes the proof. ■

LEMMA 9.6. *$f \geq 0$ integrable. Then there exits a sequence $f_n \in S(\mu)$ such that $0 \leq f_n \leq f$, f_n is increasing and*

$$I(f - f_n) \ \leq \ 4^{-n} \, .$$

PROOF. Let $\epsilon = 2^{-n}$. Furthermore, let $0 \leq \tilde{g}_n \leq f$ and $I(f) \leq I(\tilde{g}_n) + \epsilon_n$. Define $g_n = \max\{\tilde{g}_1, \dots, \tilde{g}_n\}$, then

$$I(g_n) \leq I(f) \leq I(\tilde{g}_n) + \epsilon_n \leq I(g_n) + \epsilon_n.$$

By Lemma 9.5:

$$I(g_n) + \epsilon_n \geq I(f) = I(f - g_n) + I(g_n) \, .$$

Subtracting $I(g_n)$ yields $I(f - g_n) < \varepsilon_n$. ■

LEMMA 9.7. *(Chebychev) f integrable, $\lambda > 0$. Then*

$$\lambda\mu(f \geq \lambda) \ \leq \ I(f) \, .$$

PROOF. Let $h = \lambda 1_{f \geq \lambda} \leq f$ and let $\Omega_n \subset \Omega$ an increasing sequences of subsets such that $\cup \Omega_n = \Omega$. Then

$$h_n = \lambda 1_{\{f \geq \lambda\} \cap \Omega_n} \leq I(f),$$

so

$$\mu(f \geq \lambda) = \lim_n \mu(\{f \geq \lambda\} \cap \Omega_n) = \lim_n \frac{1}{\lambda} I(h_n) \leq \frac{1}{\lambda} I(f) \, .$$ ■

LEMMA 9.8. *Let (f_n) and f be measurable such that*

$$I(|f_n - f|) \ \leq \ 4^{-n}$$

Then f_n converges to f almost everywhere.

PROOF. Let $E_n = \{\omega : |f - f_n| > 2^{-n}\}$, $F_n = \cup_{k \geq n} E_k$ and $F = \cap_n F_n$, then

$$\mu(F) \leq \lim_n \mu(F_n) \leq \lim_n \sum_{k \geq n} \mu(E_k) = \lim \sum_{k \geq n} 2^{-k} = \lim 2^{-n} = 0.$$

If $\omega \notin F \Rightarrow \exists n$ such that $\forall\ k \geq n$, $f(\omega) - 2^{-k} \leq f_k(\omega) \leq f(\omega) + 2^{-k}$, which implies

$$\lim_k f_k(\omega) = f(\omega)\ .$$

■

LEMMA 9.9. *(Beppo-Levi)* $0 \leq f_n \leq f_{n+1}$ *all integrable such that* $\lim_n I(f_n) < \infty$*. If* f_n *converges to* f *alomst everywhere, then*

$$I(f)\ \leq\ \lim_n I(f_n)\ .$$

PROOF. Let $0 \leq h \leq f$, $h \in S(\mu)$ and

$$h = \sum_{i=1}^{m} r_i 1_{E_i}$$

Let $F \subset \Omega$ such that $\lim f_n(\omega) = f(\omega)\ \forall\ \omega \in F$. Given any $\epsilon > 0$, define

$$E_{i,n} = \left\{\omega \in E_i \cap F : f_n(\omega) > \frac{r_i}{1+\epsilon}\right\} \subset E_i,$$

then $E_i \cap F = \cup_n E_{i,n}$ because $\lim_n f_n(\omega) = f(\omega) \geq r_i$. We may find n such that $\mu(E_{i,n}) \geq \frac{\mu(E_i \cap F)}{1+\epsilon}$ for all $i = 1, \ldots, m$.
Then define

$$h^{\epsilon,n} = \sum_i \frac{r_i}{1+\epsilon} 1_{E_{i,n}} \leq f_n 1_F \leq f_n\ .$$

Note that $\leq f_n 1_F \leq f_n$ and hence

$$I(h^{\epsilon,n}) \leq I(f_n) \leq \lim_n I(f_n)\ .$$

Moreover,

$$\begin{aligned} I(h) = \sum_{i=1}^{m} r_i \mu(E_i) &= \sum_{i=1}^{m} r_i \mu(E_i \cap F) \\ &\leq (1+\epsilon)^2 \sum_{i=1}^{m} \frac{r_i}{1+\epsilon} \mu(E_{i,n}) \\ &\leq (1+\epsilon)^2 I(h^{\epsilon,n}) \leq (1+\epsilon)^2 I(f_n)\ \leq\ (1+\varepsilon)^2 \lim_n I(f_n)\ . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $I(h) \leq \lim_n I(f_n)$. Taking the supremum, we deduce the assertion. ■

COROLLARY 9.10. *(Fatou) Let $0 \le f_n$ be positive integrable functions. Then*

$$I(\liminf_n f_n) \;\le\; \liminf_n I(f_n)\,.$$

PROOF. The sequence $g_n \;=\; \inf_{m \ge n} f_m$ is increasing. Then

$$I(\sup_n g_n) = \sup_n I(g_n) \;\le\; \sup_n \inf_{m\ge n} I(f_m) \;=\; \liminf_n I(f_n)\,. \qquad \blacksquare$$

PROPOSITION 9.11. *f, g positive integrable. Then*

$$I(f) + I(g) \;=\; I(f+g)\,.$$

PROOF. Let $f_n \in S(\mu)$, $g_n \in S(\mu)$ increasing sequences such that $I(f - f_n) \le 4^{-n}$ and $I(g - g_n) \le 4^{-n}$. Then the sequence $h_n = f_n + g_n$ converges to $f + g$ almost everywhere and

$$I(f+g) \le \lim_n I(f_n + g_n) \;=\; \lim_n I(f_n) + I(g_n) \;=\; I(f) + I(g)\,. \qquad \blacksquare$$

DEFINITION 9.12. *A measurable function $f : \Omega \to [-\infty, \infty]$ is called integrable if there exists a sequence (f_n) in $S(\mu)$ such that*

$$\lim_n I(|f - f_n|) \;=\; 0\,.$$

We denote $I(\mu)$ the space of integrable functions

PROPOSITION 9.13. *Let f be μ-integrable and (f_n), (f'_n) such that*

$$\lim_n I(|f - f_n|) \;=\; 0 \;=\; \lim_n I(|f - f'_n|)$$

Then

$$\lim_n I(|f_n - f'_n|) \;=\; 0\,.$$

In particular,

$$\int f \;=\; \lim_n I(f_n)$$

is well-defined.

LEMMA 9.14. *Let $f \ge 0$ be μ-integrable. Then*

$$I(f) \;=\; \int f\,.$$

PROOF. Let (f_n) be a sequence of simple functions such that $I(|f - f_n|) \leq 4^{-n}$. Then f_n converges to f almost everywhere. For fixed $n \in \mathbb{N}$ we consider $E_n = \{\omega : f_n(\omega) < 0$. Then

$$1_{E_n}|f - f_n| \;=\; 1_{E_n}|f| + 1_{E_n}|f_n| \;\geq\; 1_{E_n}|f| \,.$$

Thus we have $I(|f - f_n^+|) \;\leq\; I(|f - f_n|) \;\leq\; 4^{-n}$. Then f_n^+ converges to f μ-almost everywhere and Fatou's lemma implies

$$I(f) \;\leq\; \liminf_n I(f_n^+) \;\leq\; \sup_n I(|f_n|) \,.$$

However,

$$\begin{aligned} |I(|f_n|) - I(|f_m|)| = |I(|f_n| - |f_m|)| \;&\leq\; I(||f_n| - |f_m||) \;\leq\; I(|f_n - f_m|) \\ &\leq I(|f_n - f|) + I(|f - f_m|) \;\leq\; 4^{-n} + 4^{-m} \,. \end{aligned}$$

Thus $I(|f_n|)$ is Cauchy and hence $\sup_n I(|f_n|)$ bounded. We get

$$I(h) \;\leq\; \sup_n I(|f_n|) \,.$$

In particular, $I(f)$ is finite. Equality follows from the preceding Proposition. ■

PROPOSITION 9.15. *1) $f, g \in I(\mu)$, $\lambda \in \mathbb{R}$. Then $\int (f + \lambda g) \;=\; \int f + \lambda \int g$.*
2) $f, g \in I(\mu)$, $f \leq g$. Then $I(f) \leq I(g)$.

PROOF. 1) There exist $I(|f_n - f|) \to 0$ and $I(|g_n - g|) \to 0$, then

$$I(|f_n + \lambda g_n - (f + \lambda g)|) \leq I(|f_n - f|) + \lambda I(|g_n - g|) \to 0.$$

Since integral is well defined, so $\int f + \lambda g = \lim_n I(f_n + \lambda g_n) = I(f) + \lambda g$.
2) Consider $h = f - g \geq 0$, then $\int h = I(h) \geq 0$. Thus

$$\int f = \int (f - g + g) = \int h + \int g \geq \int g \,.$$ ■

PROPOSITION 9.16. *f is integrable iff f^+ and f^- are integrable. Moreover, we have*

$$\int f \;=\; \int f^+ - \int f^- \,.$$

PROPOSITION 9.17. *f is integrable iff f^+ and f^- are integrable. Moreover, we have*

$$\int f \;=\; \int f^+ - \int f^- \,.$$

PROOF. "$\Leftarrow$": $f = f^+ - f^-$.
"$\Rightarrow$": There exists f_n such that

$$\lim I(|f - f_n|) = 0 \Rightarrow \lim_n I(||f| - |f_n||) \Rightarrow 0,$$

so $|f|$ is integrable.

$$\int f = \int \frac{|f| + f}{2} + \int \frac{f - |f|}{2} = \int f^+ - \int f^- .$$

■

10. Convergence Theorems and applications

LEMMA 10.1. *(Fatou) Let (f_n) be positive integrable functions. Then*

$$\int \liminf_n f_n \;\le\; \liminf_n \int f_n .$$

THEOREM 10.2. *(Dominated convergence theorems) Let $f \ge 0$ be positive integrable function. Let (g_n) be integrable functions such that*

$$|g_n| \le f \quad \mu \; a.e$$

for all $n \in \mathbb{N}$ and $g = \lim_n g_n$ exists. Then g is integrable and

$$\int g \;=\; \lim_n \int g_n .$$

PROOF. Let us first assume $|g_n| \le f$ everywhere and $g = \lim_n g_n$. Then the sequence $h_n = g_n + f$ is positive. By Fatou's Lemma, we find

$$\int g + f \;=\; \int \lim_n g_n + f \;\le\; \liminf_n \int g_n + \int f .$$

Thus $g + f$ and hence g is integrable. Subtracting $\int f$ we get

$$\int g \;\le\; \liminf_n \int g_n .$$

Now, we consider $k_n = -g_n + f$ and deduce similarly as before

$$-\int g + \int f \;\le\; \liminf_n \int -g_n + \int f .$$

Thus

$$\limsup_n \int g_n \;\le\; \int g .$$

In the general case, we consider the exceptional set $E_n \in \Sigma$ of measure 0 such that $|g_n(\omega)| \le f(\omega)$ for all $\omega \in E_n^c$. Let $F \in \Sigma$ be of measure 0 such that $\lim_n g_n(\omega) = g(\omega)$ holds for $\omega \in F^c$. We define $E = F \cup \bigcup_n E_n$. Then, we have $\mu(E) = 0$. Moreover, the functions $\tilde{g_n} = g_n 1_{E^c}$ and $\tilde{g} = g 1_{E^c}$ satisfy all the requirements above. The assertion follows from the following remark. ■

REMARK 10.3. *Let E be a set of measure 0 and $f : \Omega \to 0$ be a measurable function, then $\int |f| 1_E = 0$.*

PROOF. Let $0 \le h \le |f| 1_E$. We may write

$$h \;=\; \sum_{i=1}^{m} r_i 1_{F_i} \;.$$

Then $h 1_E \;=\; h$ and hence

$$h \;=\; \sum_{i=1}^{m} r_i 1_{E\cap F_i} \;.$$

This yields $I(h) = 0$. ∎

We will now discuss an application.

THEOREM 10.4. *(Riemann-Lebesgue lemma) Let $f : \mathbb{R} \to [-\infty, \infty]$ be integrable. Then*

$$\lim_k \int \cos(kt) f(t) dt \;=\; 0 \;.$$

LEMMA 10.5. *Let f be a step function. Then*

$$\lim_k \int \cos(kt) f(t) dt \;=\; 0 \;.$$

PROOF. Let $f \;=\; \sum_{i=1}^{m} r_i 1_{[a_i,b_i]}$. By linearity it suffices to show that

$$\lim_k \int_a^b \cos(tk) dt \;=\; 0 \;.$$

This follows obviously from

$$|\int_a^b \cos(tk) dt| \;=\; \frac{|\sin(bkt) - \sin(akt)|}{k} \;\le\; \frac{2}{k} \;.$$

The assertion is proved. ∎

The Riemann-Lebsgue lemma is an easy consequence of the following result.

THEOREM 10.6. *Let $f : \mathbb{R} \to [-\infty, \infty]$ be integrable and $\varepsilon > 0$. Then there exits a simple function h such that*

$$\int |f - h| < \varepsilon \;.$$

Proof of Theorem 10.4 from Theorem 10.6. Let h be a simple function with $\int |f-h| < \frac{\varepsilon}{2}$. Let k_0 such that

$$|\int \cos(kt)h(t)dt| < \frac{\varepsilon}{2}$$

for all $k > k_0$. Then

$$\begin{aligned} |\int \cos(kt)f(t)dt| &\le |\int \cos(kt)(f(t)-h(t))dt| + |\int \cos(kt)h(t)dt| \\ &< \int |f-h| + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

holds for all $k > k_0$. ■

The following Lemma is an immediate application of the dominated convergence theorem:

LEMMA 10.7. *Let* $f : \mathbb{R} \to [-\infty, \infty]$ *be integrable and* $\varepsilon > 0$. *Then there exists* $n \in \mathbb{N}$ *such that*

$$\int_{|x|\ge n} |f| < \varepsilon \, .$$

Proof of Theorem 10.6. Let $\varepsilon > 0$. We choose $n \in \mathbb{N}$ such that

$$\int_{|x|\ge n} |f| < \frac{\varepsilon}{3} \, .$$

Let $h : [-n, n] \to \mathbb{R}$ a simple function such that

$$\int_{-n}^{n} |f-h| < \frac{\varepsilon}{3} \, .$$

Let $C = \sup |h|$ and $\delta = \frac{\varepsilon}{6(C+n)}$. We apply the consequence of Lusin's theorem and find an simple function g such that

$$m(|g-h| > \delta) < \delta \, .$$

Moreover, the construction yields such an h with $|h| \le C$. Then, we get

$$\begin{aligned} \int_{-n}^{n} |g-h| &= \int_{-n}^{n} 1_{|g-h|>\delta}|g-h| + \int_{-n}^{n} 1_{|g-h|\le\delta}|g-h| \\ &\le 2Cm(|g-h|>\delta) + 2n\delta \ \le \ 2(C+n)\delta < \frac{\varepsilon}{3} \, . \end{aligned}$$

We insist that $h = h1_{[-n,n]}$. Thus we get

$$\int |f-h| \le \int_{|x|>n} |f| + \int_{-n}^{n} |f-h| \ \le \ \int_{|x|>n} |f| + \int |f-g| + \int_{-n}^{n} |g-h| < \varepsilon. \ ■$$

LEMMA 10.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a step function and $\varepsilon > 0$. Then there exits a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} |g(x)| = 0$ and*

$$\int |f - g| < \varepsilon \, .$$

PROOF. Let $f = 1_{[a,b]}$ and $0 < \delta < b - a$. Then

$$g_{\delta,a,b}(t) \;=\; \begin{cases} \delta^{-1}(t-(a-\delta)) & \text{if } a-\delta \le t \le a \\ 1 & \text{if } a \le t \le b \\ 1-\delta^{-1}(t-b) & \text{if } b \le t \le b+\delta \\ 0 & \text{else} \end{cases} \, .$$

Then g is continuous and

$$\int |g_{\delta,a,b} - 1_{[a,b]}| d\mu \le 2 \int_0^\delta t dt \;=\; \delta \, .$$

For an arbitrary simple function $f = \sum_{i=1}^n r_i 1_{[a_i,b_i]}$ we consider $g_\delta \;=\; \sum\limits_{i=1}^n r_i g_{\delta,a_i,b_i}$. Then we get

$$\int |f - g_\delta| \le \sum_{i=1}^n |r_i| \int |1_{[a,b_i]} - g_{\delta,a_i,b_i}(t)| \;\le\; \sum_{i=1}^n |r_i| \delta \, .$$

Thus $\delta < \frac{\varepsilon}{1+\sum\limits_{i=1}^n |r_i}$ implies the assertion. ■

COROLLARY 10.9. *Let $f : \mathbb{R} \to [-\infty, \infty]$ be an integrabe function and $\varepsilon > 0$. Then there exits a continuous function g vanishing at $\pm\infty$ such that*

$$\int |f - g| d\mu < \varepsilon \, .$$

PROOF. Let h be a step function such that

$$\int |f - h| d\mu < \frac{\varepsilon}{2} \, .$$

Let g be a continuous function (constructed above) such that $\int |g - h| < \frac{\varepsilon}{2}$. This function g vanishes for large x's and satisfies

$$\int |f - g| d\mu \;\le\; \int |f - h| d\mu + \int |h - g| < \varepsilon \, .$$

This proves the assertion. ■

11. Banach spaces

DEFINITION 11.1. *A normed space is given by a vector space V (over $K = \mathbb{R}$ or $K = \mathbb{C}$) and a function $\| \; \| : V \to [0, \infty)$ satisfying the following conditions*

i) $\|x\| = 0 \Leftrightarrow x = 0$,
ii) $\|\lambda x\| = |\lambda| \|x\|$,
iii) $\|x + y\| \leq \|x\| + \|y\|$,

for all $x, y \in V$, $\lambda \in K$. The associated metric on $(V, \| \; \|)$ is defined by

$$d_{\| \; \|}(x, y) \; = \; \|x - y\| \, .$$

REMARK 11.2. *$+ : V \times V \to V$ given by $+(x, y) = x + y$ and $\cdot : K \times V \to V$ given by $\cdot(\lambda, x) = \lambda x$ are continuous. Moreover, $\| \; \| : V \to [0, \infty)$ is continuous.*

In the following we will mostly consider real vector spaces (because the name of our course is real analysis).

DEFINITION 11.3. *A Banach space is a normed vector space such that $(V, d_{\| \; \|})$ is complete.*

EXAMPLE 11.4. (1) *On $V = \mathbb{R}^n$ we define*

$$\|x\|_p \; = \; \left(\sum_{i=1}^{n} \|x_i|^p \right)^{\frac{1}{p}}$$

and $\|x\|_\infty = \max_{i=1,\ldots,n} \|x_i|$. Then $(\mathbb{R}^n, \| \; \|_p)$ is Banach space (see below for the triangle inequality).

(2) *$\ell_p = \{(x_n) \; : \; \sum_n |x_n|^p < \infty\}$ is a Banach space with respect to*

$$\|(x_n)\|_p \; = \; \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{\frac{1}{p}} \, .$$

(3) *If $\| \; \|$ is a norm on $\mathbb{R}^n$, then $(\mathbb{R}^n, \| \; \|)$ is a Banach space.*

(4) *$(C[0, 1], \| \; \|_1)$ where*

$$\|f\|_1 \; = \; \int_0^1 |f(s)| ds$$

is a normed space, but not a Banach space.

PROPOSITION 11.5. *Let X be a normed space and Y be a Banach space. We define $L(X, Y)$ as the space of map $T : X \to Y$ which are linear, i.e.*

$$T(x + \lambda y) \; = \; T(x) + \lambda T(y) \, .$$

and continuous. The norm on $L(X,Y)$ is given by

$$\|T\|_{op} \;=\; \sup_{\|x\|\leq 1} \|T(x)\| \,.$$

Then $L(X,Y)$ is a Banach space.

Proof. Let us first show that a linear map $T : X \to Y$ is continuous iff $\|T\| < \infty$. Indeed, if $\|T\|$ is finite, then

$$\|T(x) - T(y)\| \;=\; \|T(x-y)\| \;\leq\; \|T\|_{op}\|x-y\|$$

holds for all $x, y \in V$. Thus T is Lipschitz and thus continuous. For the converse, we assume that T is continuous. Then $T^{-1}(B(0,1))$ is open and henceforth contains $B(0,\varepsilon)$ for some $\varepsilon > 0$. Now let $\|x\| \leq 1$ and $0 < \delta < \varepsilon$. Then $\|(\varepsilon-\delta)x\| < \varepsilon$ and hence

$$\|T(x)\| \;=\; (\varepsilon-\delta)^{-1}\|T(\varepsilon-\delta)(x)\| < (\varepsilon-\delta)^{-1} \,.$$

This shows that $\|T\|_{op} \leq (\varepsilon-\delta)^{-1}$ for every $\delta > 0$ and thus $\|T\|_{op} \leq \varepsilon^{-1}$. Now, we observe that $\| \, \|_{op}$ is a norm. We only check the triangle inequality. Indeed,

$$\begin{aligned}\|T+S\|_{op} &= \sup_{\|x\|\leq 1} \|(T+S)(x)\| \;=\; \sup_{\|x\|\leq 1} \|T(x)+S(x)\| \;\leq\; \sup_{\|x\|\leq 1} \|T(x)\| + \|S(x)\| \\ &\leq \|T\|_{op} + \|S\|_{op} \,.\end{aligned}$$

Finally we have to show that $L(X,Y)$ is complete. Let (T_n) be a Cauchy sequence of linear maps. For fixed $x \in X$, we have

$$\|T_n(x) - T_m(x)\| \;\leq\; \|T_n - T_m\|\|x\| \,.$$

Thus $(T_n(x))$ is Cauchy and we may define

$$T(x) \;=\; \lim_n T_n(x) \,.$$

Then we have

$$T(x+\lambda y) \;=\; \lim_n T_n(x+\lambda y) \;=\; \lim_n T_n(x) + \alpha T_n(y) \;=\; T(x) + \lambda T(y) \,.$$

Thus T is linear. Let us show that

$$\lim_n \|T - T_n\|_{op} \;=\; 0 \,. \tag{11.1}$$

Indeed, let $x \in X$ with $\|x\| \leq 1$. Then we have

$$\begin{aligned}\|T(x) - T_n(x)\| &= \|\lim_m T_m(x) - T_n(x)\| \;\leq\; \limsup_{m\geq n} \|T_m(x) - T_n(x)\| \\ &\leq \sup_{m\geq n} \|T_m - T_n\|\|x\| \;\leq\; \sup_{m\geq n} \|T_m - T_n\| \,.\end{aligned}$$

In particular $\|T\|_{op} \leq \|T - T_1\|_{op} + \|T_1\|_{op}$ is finite and T is continuous. Moreover, $\lim_n d(T, T_n) = 0$ implies that $\lim_n T_n = T$. ■

COROLLARY 11.6. *Let X be a normed space. Then $X^* = L(X, \mathbb{R})$ is a Banach space. Moreover, $X^{**} = L(X, \mathbb{R})$ is a Banach space.*

DEFINITION AND REMARK 11.7. *Let $\iota : X \to X^{**}$ be the linear map given by $\iota(x)(x^*) = x^*(x)$. Then*

$$\|\iota(x)\| \leq \|x\| .$$

Indeed, the Hahn-Banach theorem (proved in the next course) shows that $\|\iota(x)\| = \|x\|$. A Banach space X is called reflexive if $\iota(X) = X^{**}$, i.e. ι is surjective. All finite dimensional spaces are reflexive.

12. L_p spaces

In the following (Ω, Σ, μ) is a sigma-finite measure space. We define

$$\mathcal{L}_0 \;=\; \{f : \Omega \to \mathbb{R} \;:\; \lim_{\alpha\to\infty} \mu(|f| > \alpha) \;=\; 0\} \,.$$

On $\mathcal{L}_0$ we define the equivalence relation

$$f \sim g \quad \text{if } f \;=\; g\; \mu \; a.e.$$

i.e. there exists a set $F \in \Sigma$ with measure 0 such that $f(\omega) = g(\omega)$ for all $\omega \in F^c$. We define

$$L_0(\mu) \;=\; \mathcal{L}_0/\sim$$

Proposition 12.1. *(Hw) $L_0(\mu)$ equipped with the distance*

$$d([f],[g]) \;=\; \inf\{\varepsilon \;:\; \mu(|f-g| > \varepsilon) < \varepsilon\}$$

is a complete metric space.

Definition 12.2. *$\mathcal{L}_p$ is the set of all measurable functions f such that $\int |f|^p$ is finite.*

Lemma 12.3. *(Hölder's inequality) Let $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in \mathcal{L}_p$, $g \in \mathcal{L}_q$. Then fg is integrable and*

$$|\int fg| \;\le\; (\int |f|^p)^{\frac{1}{p}} (\int |q|^q)^{\frac{1}{q}} \,.$$

Proof. We use the fact that $g(x) = -\ln(x)$ is convex. Thus for positive numbers a, b we have

$$-\ln(\frac{a^p}{p} + \frac{b^q}{q}) \;=\; g(\frac{a^p}{p} + \frac{b^q}{q}) \;\le\; \frac{1}{p} g(a^p) + \frac{1}{q} g(b^q) \;=\; -\ln(a) - \ln(b) \,.$$

This yields

$$ab \;\le\; \frac{a^p}{p} + \frac{b^q}{q} \,. \tag{12.1}$$

Thus for every $\omega \in \Omega$ and $s > 0$ we find

$$|f(\omega)||g(\omega)| \;\le\; \frac{|sf(\omega)|^p}{p} + \frac{|s^{-1}g(\omega)|^q}{q} \,.$$

Using this for $s = 1$, we deduce that $\int |f|^p < \infty$ and $\int |g|^q < \infty$ implies $\int |fg| < \infty$. Thus we get

$$|\int fg| \;\le\; \int |fg| \;\le\; \frac{s^p}{p} \int |f|^p + \frac{s^{-q}}{q} \int |g|^q \,.$$

We define $s = \frac{(\int |g|^q)^{1/(p+q)}}{(\int |f|^p)^{1/(p+q)}}$. This implies

$$s^p \int |f|^p \;=\; s^{-q} \int |g|^q \,.$$

Hence, we deduce from $p/(p+q) = 1/q$ and $q/(p+q) = 1/p$ that

$$\begin{aligned} |\int fg| \le s^p \int |f|^p \;&=\; (\int |g|^q)^{p/(p+q)} (\int |f|^p)^{1-p/(p+q)} \\ &= (\int |g|^q)^{1/q} (\int |f|^p)^{1/p} \,. \end{aligned}$$

■

REMARK 12.4. *Let f be a measurable function. Then*

$$(\int |f|^p)^{1/p} \;=\; \sup\{ |\int fg| \;:\; g \in S(\mu) \int |g|^q \le 1 \} \,.$$

PROOF. Let $0 \le h \le |f|^p$ be a simple function. This implies $0 \le h^{1/p} \le |f|$. We write $h^{1/p} \;=\; \sum_i r_i 1_{E_i}$ and define

$$g(\omega) \;=\; (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} 1_{E_i(\omega)} \frac{f(\omega)}{|f(\omega)|} \,.$$

Then, we get

$$\begin{aligned} \int fg \;&=\; (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} \int_{E_i} |f(\omega)| \\ &\ge\; (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} \int_{E_i} r_i \\ &= (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^{\frac{p}{q}} r_i \mu(E_i) \\ &= (\int |h|)^{-\frac{1}{q}} \sum_{i=1}^n r_i^p \mu(E_i) \;=\; (\int h)^{\frac{1}{p}} \,. \end{aligned}$$

On the other hand

$$\int |g|^q d\mu = (\int |h|)^{-1} \sum_i r_i^p \mu(E_i) \;=\; 1 \,.$$

This yields the assertion. ■

LEMMA 12.5. *Let f and g be measurable functions. Then*

$$(\int |f+g|^p)^{\frac{1}{p}} \;\le\; (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} \,.$$

PROOF. It suffices to consider $1 < p < \infty$. We may assume that the right hand is finite. Let $0 \le h \le |f+g|$. Then, we have

$$h^p \;=\; hh^{p-1} \;\le\; (|f|+|g|)h^{p-1}\,.$$

By Lemma 12.3 we deduce (for $\frac{1}{q} = 1 - \frac{1}{p}$)

$$\begin{aligned}\int h^p &\le \int |f|h^{p-1} + \int |g|h^{p-1}\\ &\le (\int |f|^p)^{\frac{1}{p}}(\int h^{(p-1)q})^{1-\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}}(\int h^{(p-1)q})^{1-\frac{1}{p}}\\ &= \Big((\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}}\Big)(\int h^p)^{1-\frac{1}{p}}\,.\end{aligned}$$

If $\int h^p = 0$ there is nothing to show. In the other case we obtain

$$(\int h^p)^{\frac{1}{p}} \;\le\; (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}}\,.$$

Taking the sup over all $0 \le h \le |f+q|$ we deduce the assertion. ■

THEOREM 12.6. *Let $1 \le p < \infty$. The space*

$$L_p \;=\; \{[f]\;:\; f \textit{ measurable }\;, \|[f]\|_p \;=\; (\int |f|^p)^{\frac{1}{p}}\}$$

with the norm $\| \; \|_p$ is a Banach space.

PROOF. We note first that for $f \sim g$ we have

$$\int |f|^p \;=\; \int |g|^p\,.$$

Thus $\| \; \|_p$ is well-defined. Moreover, we have

$$\|[\lambda f]\|_p \;=\; |\lambda|\|[f]\|_p\,.$$

By the definition of equivalent classes, we see that

$$\|[f]\|_p \;=\; 0 \Leftrightarrow f = 0 \; a.e. \Leftrightarrow [f] = 0\,.$$

For $x, y \in L_p$, we pick $f \in x$, $g \in y$. According to Lemma 12.5, we deduce that $|f+g|^p$ is integrable and hence $x + y = [f+g]$ is in L_p satisfying

$$\|x+y\|_p \;=\; \|[f+g]\|_p \;=\; (\int |f+g|^p)^{\frac{1}{p}} \;\le\; (\int |f|^p)^{\frac{1}{p}} + (\int |g|^p)^{\frac{1}{p}} \;=\; \|x\|_p + \|y\|_p\,.$$

Thus $(L_p, \| \; \|_p)$ is a normed vector space. Let (x_n) be a Cauchy sequence in L_p. We may assume

$$\|x_n - x_{n+1}\|_p \;\le\; 2^{-n\frac{p+1}{p}}\,.$$

Let $f_n \in x_n$. Then we find

$$\int |f_n - f_{n+1}|^p \;\le\; 2^{-n(p+1)} \,.$$

By Chebychev's inequality we deduce

$$\mu(|f_n - f_{n+1}| > 2^{-n}) 2^{-np} \;\le\; \int |f_n - f_{n+1}|^p \;\le\; 2^{-n(p+1)} \,.$$

Thus we get

$$\mu(|f_n - f_{n+1}| > 2^{-n}) \;\le\; 2^{-n} \,.$$

The convergence Lemma implies that (f_n) is almost everywhere convergent to a measurable function f. Define

$$h \;=\; |f_1| + \sum_n |f_{n+1} - f_n| \,.$$

We want to show that $|h|^p$ is integrable. Indeed, we apply the monotone convergence Lemma and deduce from the triangle inequality in $\mathcal{L}_p$

$$\begin{aligned} \int |h|^p &\le \liminf_m \int (|f_1| + \sum_{n=1}^m |f_{n+1} - f_n|)^p \\ &\le \liminf_m (\|[f_1]\|_p + \sum_{n=1}^m \|[f_{n+1} - f_n\|_p)^p \\ &\le (\|x_1\|_p + \sum_n \|x_{n+1} - x_n\|_p)^p \;\le\; (\|x_1\|_p + 2)^p \;<\infty \,. \end{aligned}$$

Moreover, we have $|f_n - f_m|^p \;\le\; |h|^p$ and for $m \ge n$ we get that

$$(\int |f_m - f_n|^p)^{\frac{1}{p}} \;\le\; \sum_{k=n}^m \|x_{k+1} - x_k\|_p \;\le\; \sum_{k=n}^{\infty} 2^{-k\frac{p+1}{p}} \;\le\; 2^{1-n} \,.$$

By the dominated convergence theorem (with majorant $|h|^p$) we deduce that

$$\int |f - f_n|^p d\mu \;=\; \lim_{m \ge n} \int |f_m - f_n|^p \;\le\; 2^{p(1-n)} \,.$$

Using the triangle and $f_1 \in \mathcal{L}_p$, we deduce that $f \in L_p$. Moreover,

$$\lim_n \|[f] - x_n\|_p \;=\; 0 \,.$$

This completes the proof. ∎

Proposition 12.7. *Let $1 \le p < \infty$. The simple functions are dense in L_p. For $(\Omega, \Sigma, \mu) = (\mathbb{R}, \mathcal{L}, m)$ the step functions are dense in L_p and the continuous functions are dense in L_p.*

PROOF. Let $f \geq 0$ such that $\int f^p < \infty$. Let h_n be an increasing sequence of simple functions such that

$$I(f^p - h_n) < 4^{-n} .$$

Then h_n converges to f^p a.e. and also $h^{\frac{1}{p}}$ converges to f almost everywhere. Since $|f - h_n|^p \;\leq\; |f|^p$, we deduce from the dominated convergence theorem that

$$\lim_n \int |f - h_n|^p \;=\; \int 0 \;=\; 0 .$$

This proves $\lim_n \|[f] - [h_n]\|_p \;=\; 0$. The general assertion follows by considering $f = f^+ - f^-$. For the second assertion, we assume again that $f \geq 0$ and $0 \leq h \leq f$ such that

$$\int |f - h|^p < \varepsilon .$$

Let $C = \sup \|h\|$. Using a small perturbation, we may also assume that h vanishes in $(-\infty, n] \cup [n, \infty)$. By the applications of Lusin's theorem, we may find a step function g such that $0 \leq g \leq C$ and

$$mu(|g - h| > \delta) < \delta .$$

Then, we get

$$\begin{aligned} \int_{-n}^{n} |g - h|^p &= \int_{-n}^{n} 1_{|g-h|>\delta} |g - h|^p + \int_{-n}^{n} 1_{|g-h|\leq\delta} |g - h|^p \\ &\leq (2C)^p m(|g - h| > \delta) + \delta^p 2n \;\leq\; (2C)^p \delta + 2n\delta^p . \end{aligned}$$

Choosing δ small enough we get $(\int_{-n}^{n} |g - h|^p)^{\frac{1}{p}} < \varepsilon$. Starting from step functions, continuous functions are achieved as in Lemma 2.8. ∎

13. Hilbert spaces

A (real) Hilbert space comes with a vector space H, a scalar product $(\ ,\ ) : H \times H \to \mathbb{R}$ satisfying

i) $(x,x) \geq 0$ and $(x,x) = 0$ iff $x = 0$,
ii) $(x,y) = (y,x)$ and
iii) $(x, y + \lambda z) = (x,y) + \lambda(x,z)$,

for all $x, y, z \in H$ and $\lambda \in \mathbb{R}$. Moreover, we require that H is complete with respect to the metric

$$d(x,y) = (x-y, x-y)^{\frac{1}{2}} .$$

(With completeness this is called a pre-Hilbert space.)

Lemma 13.1. $\|x\| = (x,x)^{\frac{1}{2}}$ *is a norm. Moreover,*

$$|(x,y)| \leq \|x\| \|y\| . \tag{13.1}$$

Proof. We first show

$$|(x,y)| \leq \|x\| \|y\| .$$

For this let $\lambda \in \mathbb{R}$. Then we have

$$0 \leq (x + \lambda y, x + \lambda y) = (x,x) + 2\lambda(x,y) + \lambda^2(y,y) .$$

This yields

$$2(-\lambda)(x,y) \leq (x,x) + \lambda^2(y,y) .$$

If $(x,y) \geq 0$ we define $\lambda = -\frac{\|x\|+\varepsilon}{\|y\|+\varepsilon}$. This yields

$$\begin{aligned} 2|(x,y)| &\leq \frac{\|y\|+\varepsilon}{\|x\|+\varepsilon}(\|x\|^2 + (\|x\|+\varepsilon)^2 \frac{\|y\|^2}{(\|y\|+\varepsilon)^2} \leq \frac{\|y\|+\varepsilon}{\|x\|+\varepsilon} 2(\|x\|+\varepsilon)^2 \\ &\leq 2(\|y\|+\varepsilon)(\|x\|+\varepsilon) . \end{aligned}$$

Letting $\varepsilon \to 0$ yields (13.1). Now, we consider again $x, y \in H$. Then

$$\begin{aligned} \|x+y\|^2 = (x+y, x+y) &= (x,x) + 2(x,y) + (y,y) \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2 . \end{aligned}$$

■

Definition 13.2. *A system* (x_i) *is called an orthonormal if*

$$(x_i, x_j) = \delta_{ij} .$$

A system $(x_i) \subset H$ *is called orthonormal basis if moreover,*

$$\overline{\{\sum_i \lambda_i x_i \ : \ \lambda_i \in \mathbb{R}\}} \ = \ H \ .$$

Here we are taking only finite linear combinations.

LEMMA 13.3. *Let* (x_i) *be an orthonormal system and* $x \in H$*. Then*

$$(\sum_i |(x_i,x)|^2)^{\frac{1}{2}} \ \le \ \|x\| \ .$$

Moreover, $y = \sum_i (x_i,x)x_i$ *is in the closure of* $\{\sum_i \lambda_i x_i\}$.

PROOF. Let $J \subset I$ be a finite subset. Let $\lambda_i \ = \ (x_i,x)$ and consider

$$y \ = \ \sum_{i\in J} \lambda_i x_i \ .$$

Then, we have

$$(y,y) \ = \ \sum_{i,j} \lambda_i \lambda_j (x_j,x_i) \ = \ \sum_j |\lambda_j|^2 \ .$$

By (13.1) we deduce

$$\sum_{j\in J} |(x_i,x)|^2 = |(x,y)| \ \le \ \|x\|(\sum_i \|(x_i,x)|^2)^{\frac{1}{2}} \ .$$

Cancellation yields

$$(\sum_{i\in J} |(x_i,x)|^2)^{\frac{1}{2}} \|x\| \ .$$

Note that

$$\sum_i |(x_i,x)|^2 \ = \ \sup_{J\subset I finite} \sum_{i\in J} |(x_i,x)|^2$$

is the considered as a definition here. For the second assertion, we consider $I_n = \{i \in I \, : \, |(x_i,x)| > \frac{1}{n}\}$. Note that I_n has to be finite set. Thus $I' = \bigcup_n I_n$ is countable set. We may assume $I' = \mathbb{N}$ and

$$\sum_n |(x_{i_n},x)|^2 < \infty \ .$$

Thus for every $\varepsilon > 0$ we may find n_0 such that for every $m > n > n_0$ we have

$$\sum_{k=n}^{m} |(x_{i_k},x)|^2 < \varepsilon \ .$$

By the above, we deduce that

$$y_n \ = \ \sum_{k=1}^{n} (x_{i_k},x) x_{i_k}$$

is Cauchy and that the limit y satisfies $(x_i, y) = (x_i, x)$ and

$$\lim_n \|y - y_n\| \;=\; 0 \,.$$

This complete the proof. ■

REMARK 13.4. *The element* $y = \sum_i (x_i, x) x_i$ *satisfies*

$$\inf_{z \in cl(\{\sum_i \lambda_i x_i\})} \|x - z\| \;=\; \|x - y\| \,.$$

LEMMA 13.5. *Let* (x_i) *be orthonormal. Then there exists an orthonormal basis containing* (x_i)*.*

PROOF. Let S be the collection of orthonormal sets containing (x_i). It is easily checked that for every chain the union is an element in S. By Zorn's Lemma we may find a maximal element (y_j) in S. Let us assume that $x \in H$ does no belong to the closure of $\{\sum_j \lambda_j y_j\}$. Then, we define

$$z \;=\; x - \sum_j (y_j, x) y_j$$

and $y = z/\|z\|$. It is easily seen that y has norm 1 and $(y, y_j) = 0$ for all j. Thus we may add y to the system (y_j). This contradiction concludes the proof. ■

COROLLARY 13.6. *The dual of* H *is* H *and* H *is reflexive.*

PROOF. Let $f : H \to \mathbb{C}$ be a linear functional. Let (x_i) be an ONS. We define

$$\lambda_i \;=\; f(x_i) \,.$$

Let $J \in I$ be a finite subset. Then

$$\begin{aligned} \sum_{i \in J} |\lambda_i|^2 = |f(\sum_{i \in J} \lambda_i x_i)| \;&\le\; \|f\| \|\sum_i \lambda_i x_i\| \\ &\le \|f\| (\sum_i \|\lambda_i\|^2)^{\frac{1}{2}} \,. \end{aligned}$$

Thus we deduce

$$(\sum_{i \in I} |\lambda_i|^2)^{\frac{1}{2}} \;\le\; \|f\| \,.$$

We have seen in Lemma 13.3 that $y = \sum_i \lambda_i x_i$ converges and hence

$$(y, x_i) \;=\; \lambda_i \,.$$

Thus f and the functional $f_y(x) = (y, x)$ coincide on a dense set and are Lipschitz. By the unique extension principle they coincide. Thus the dual H^* of H is exactly given by functionals $f_y(x) = (y, x)$ and $\|f_y\| = \|y\|$. Hence the dual of H^* is given by functionals $g_z(f_y) = (z, y)$, $z \in H$. This implies $H^{**} = H$. ■

COROLLARY 13.7. *The dual of* $L_2(\Omega, \Sigma, \mu)$ *is* $L_2(\Omega, \Sigma, \mu)$.

PROOF. We note that

$$(f, g) \;=\; \int fg$$

is a scalar product and $\|f\|_2 = (f, f)^{\frac{1}{2}}$. Since $L_2(\Omega, \Sigma, \mu)$ is complete we deduce that $L_2(\Omega, \Sigma, \mu)$ is Hilbert space. Thus the dual is given by the linear functionals $\phi_f : L_2(\Omega, \Sigma, \mu) \to \mathbb{R}$ defined as

$$\phi_f(g) \;=\; \int fg\,.$$

This yields the assertion. ■

For the next duality result we will have to introduce the space $L_\infty(\Omega, \Sigma, \mu)$ of equivalence class with the norm

$$\|f\|_\infty \;=\; \inf_{\mu(E)=0} \sup_{\omega \in E^c} |f(\omega)|\,.$$

LEMMA 13.8. *Let f be a measurable function. Then*

$$\|f\|_\infty \;=\; \sup_{\int |g| \le 1} |\int fg|\,.$$

PROOF. Let $\|f\|_\infty < 1$. Then there exists a set E of measure 0 such that $|f(\omega)| \le 1$ for all $\omega in E^c$. Thus we get by monotonicity

$$|\int fg| \;\le\; \int_{E^c} |f||g| \;\le\; \int_{E^c} |g| \;\le\; \int |g| d\mu\,.$$

For the converse we assume $\|f\|_\infty > r$. Let $E = \{\omega : |f| > r\}$. Then we must $\mu(E) > 0$. Since Ω is σ-finite we find a subset $F \subset E$ with $0 < \mu(F) < \infty$. We define

$$g(\omega) \;=\; \mu(F)^{-1} 1_F \frac{f(\omega)}{|f(\omega)|}\,.$$

Then $\int |g| \;=\; 1$ and

$$\int fg \;=\; \mu(F)^{-1} \int_F |f| \;\ge\; r\,.$$

This yields the assertion. ■

COROLLARY 13.9. *Let* $1 \le p \le 2$ *and* (Ω, Σ, μ) *be probability space. Then the dual space of* $L_p(\Omega, \Sigma, \mu)$ *is* $L_{p'}(\Omega, \Sigma, \mu)$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$.

PROOF. Let us first consider a function $f \in L_2$. We define $r = 2/p \geq 1$. Then Hölder's inequality implies

$$\int |f|^p d\mu = (\int |f|^{pr})^{\frac{1}{r}} (\int 1 d\mu)^{1-\frac{1}{r}} \; = \; (\int |f|^2)^{\frac{1}{r}} \; .$$

This implies

$$\|f\|_p \; \leq \; \|f\|_2 \; .$$

Now, let $\phi : L_p \to \mathbb{R}$ be a continuous linear functional. Then ϕ is a continuous linear functional on L_2. Thus we can find a function $g \in L_2$ such that

$$\phi(f) \; = \; \int g f d\mu$$

holds for all $g \in L_2$. Let h be a measurable function such that $|h|$ is a simple function. Then $h \in L_2$ and hence Remark 1.9. implies

$$(\int \|g|^{p'} d\mu)^{\frac{1}{p'}} \; = \; \sup_{|h| \in S(\mu) \|h\|_p \leq 1} |\int g h d\mu| \; \leq \; \|\phi\| \; .$$

Thus $[g] \in L_{p'}(\Omega, \sigma, \mu)$. By Hölder's inequality we deduce

$$\phi_g(f) \; = \; \int g f d\mu$$

is a continuous linear functional. Moreover, for every simple function h we deduce from $h \in L_2$ that

$$\phi(h) \; = \; \phi_g(h) \; .$$

By density of the simple functions (and the unique extension principle) ϕ and ϕ_g coincide. ■

REMARK 13.10. *The previous result extends easily to σ-finite measure spaces and is true in full generality.*

14. Radon-Nikodym Theorem

DEFINITION 14.1. *Let Σ be a σ-algebra and ν and μ measures on Σ. We say that ν is absolutely continuous with respect to μ (in short $\nu \ll \mu$) if $\mu(E) = 0$ implies $\nu(E) = 0$.*

THEOREM 14.2. *Let μ and ν be σ-finite measure such that $\nu \ll \mu$. Then there exists a positive measurable function g such that*

$$\nu(E) \;=\; \int_E g d\mu \,.$$

PROOF. We consider the measure $\mu_1 \;=\; \mu+\nu$. It is easily check that μ_1 is also σ-finite. Therefore we assume $\mu_1(\Omega) < \infty$. Then the linear functional $\phi : L_1(\mu_1) \to \mathbb{R}$ defined by

$$\phi(f) \;=\; \int f d\nu \,.$$

For a positive simple function $f = \sum_i a_i 1_{E_i}$ we have

$$\phi(f) \;=\; \sum_i a_i \nu(E_i) \;\le\; \sum_i a_i \mu_1(E_i) \;=\; \|f\|_1 \,.$$

By density ϕ extends to a function of norm ≤ 1. According to Corollary 13.9 we find $g \in L_\infty(\mu_1)$ such that

$$\phi(f) \;=\; \int g f d\mu_1 \,.$$

It is easily checked that g is positive and hence $0 \le g \le 1$. Thus, we deduce inductively

$$\begin{aligned} \int f d\nu &= \int f g d\mu + \int f g d\nu \\ &= \int f g d\mu + \int f g^2 d\mu + \int f g^2 d\nu \\ &= \int f g d\mu + \int f g^2 d\mu + \int f g^3 d\mu + \int f g^3 d\nu \\ &= \int f (\sum_{k=1}^n g^k) d\mu + \int f g^n d\nu \,. \end{aligned} \tag{14.1}$$

Let us consider the set $E_m = \{\omega : 0 \le g(\omega) < 1 - \frac{1}{m}\}$. Passing to the limit we deduce that

$$\int_{E_m} f d\mu \;=\; \lim_n \int_{E_m} f (\sum_{k=1}^n g^k) d\mu + \int f g^n d\nu \;=\; \int_{E_m} f g (1-g)^{-1} d\mu \,.$$

Let $E = \bigcup_m E_m$. By the dominated convergence theorem we deduce that for every $F \subset \Omega$ we have

$$\int_{F\cap E} d\nu \;=\; \lim_m \int_{F\cap E_m} d\nu \;=\; \lim_m \int_{F\cap E_m} (1-g)^{-1} d\mu \;=\; \int_{E\cap F} g(1-g)^{-1} d\mu \,.$$

Let us consider the set E^c where $g = 1$. Then we deduce from (14.1) that

$$\int_E d\nu \;=\; \int_E d\mu + \int_E d\nu$$

Thus $\mu(E) = 0$. By absolute continuity we get $\nu(E) = 0$. Therefore

$$\int_F d\nu \;=\; \int_F 1_E g(1-g)^{-1} d\mu$$

holds for every $F \in \Sigma$. ■

COROLLARY 14.3. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'}$. The dual space of L_p is $L_{p'}$.*

PROOF. Let us assume μ finite. Let $\phi : L_p(\mu) \to \mathbb{R}$ be a continuous linear functional. We define

$$\nu(E) \;=\; \sup\{\sum_j |\phi(1_{E_j})| \;:\; E \;=\; \dot{\bigcup}_j E_j \;,\; E_j's \text{ disjoint } \} \,.$$

Clearly, $E_1 \subset E_2$ implies $\nu(E_1) \le \nu(E_2)$. Moreover, let $E = \dot{\bigcup}_j E_j$ and $\varepsilon_j = \frac{\phi(1_{E_j})}{|\phi(1_{E_j})|}$. Note that

$$h_n \;=\; \sum_{j\le n} \varepsilon_j 1_{E_j}$$

satisfies $|h_n| \le 1_\Omega$ and hence the dominated convergence theorem implies

$$\lim_n \|\sum_{j\ge n} \varepsilon_j 1_{E_j}\|_p \;=\; 0 \,.$$

Since ϕ is continuous we deduce

$$\sum_j |\phi(E_j)| \;=\; |\phi(\sum_j \varepsilon_j 1_{E_j})| \;\le\; \|\phi\|\|\sum_j \varepsilon_j 1_{E_j}\|_p \;\le\; \|\phi\|\mu(E)^{\frac{1}{p}} \,.$$

Thus $\nu(E) \le \mu(E)^{\frac{1}{p}}$. Let us show that ν is σ-additive. Indeed, let (F_i) be disjoint and $F_i \;=\; \cdot\bigcup E_{ij}$. Then

$$\nu(\bigcup_j F_j) \;\ge\; \sum_{i,j} |\phi(E_{ij})| \,.$$

Taking the supremum we get

$$\nu(\bigcup_j F_j) \;\ge\; \sum_i \nu(E_i) \,.$$

Conversely $\bigcup_j F_j = \bigcup_i E_i$. Then $F_j = \dot{\bigcup}_i F_j \cap E_i$. And hence

$$\sum_i |\phi(1_{E_i})| \le \sum_{i,j} |\phi(1_{E_i \cap F_j})| \le \sum_j \nu(F_j) .$$

Thus ν is a σ-additive measure which is absolutely continuous with respect to μ. This we find a positive function g such that

$$\nu(E) = \int g d\mu .$$

Now, we consider $\Phi : L_1(\nu) \to \mathbb{R}$ defined by

$$\Phi(\sum_i a_i 1_{E_i}) = \sum_i a_i \phi(1_{E_i}) .$$

Then

$$|\Phi(\sum_i a_i 1_{E_i})| \le \sum_i |a_i| \nu(E_i)$$

Thus ϕ is a continuous functional and hence we find a measurable function h such that

$$\phi(\sum_i a_i 1_{E_i}) = \sum_i \int_{E_i} a_i h d\nu = \sum_i \int_{E_i} a_i hg d\mu = \int (\sum_i a_i 1_{E_i}) hg d\mu .$$

Since the simple functions are dense, we deduce from Remark 1.9. that $\|hg\|_{p'} \le \|\phi\|$ and

$$\phi(f) = \int f hg d\mu .$$

This assertion is proved. ■

COROLLARY 14.4. *Let $1 < p < \infty$. Then L_p is reflexive.*

PROOF. Indeed, $L_p^{**} = L_{p'}^{*} = L_p$. ■

15. Absolute continuous functions

A function $f : [a, b] \to \mathbb{R}$ is called of bounded variation if

$$\|f\|_{BV} \;=\; \sup\{\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \;:\; a = x_0 < x_1 < \cdots < x_n = b\}$$

is finite. We say that f is absolutely continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition $a = x_0 < x_1 < \cdots < x_n = b$ and every subset $J \subset \{1, ..., n\}$

$$\sum_{i \in J} |x_{i+1} - x_i| < \delta \Longrightarrow \sum_{i \in J} |f(x_{i+1}) - f(x_i)| < \varepsilon \,.$$

Lemma 15.1. *Let $f \in L_1[a, b]$ and $F(t) = \int\limits_a^t f(s)dm(s)$. Then F is of bounded variation and absolutely continuous. Moreover, $F(a) = 0$ and $\|F\|_{BV} \leq \int |f|$.*

Proof. For a partition $a = x_0 < x_1 < \cdots < x_n = b$ and $J \subset \{1, .., n\}$ and $\varepsilon_i = \frac{F(x_{i+1} - F(x_i))}{|F(x_{i+1} - F(x_i))|}$ we have

$$\sum_{i \in J} |F(x_{i+1} - F(x_i))| = |\int (\sum_{i \in J} \varepsilon_i 1_{[x_i, x_{i+1}]}) f dm| \;\leq\; \int |f| 1_{\bigcup_{i \in J} [x_i, x_{i+1}]} dm \,.$$

Thus for $J = \{1, .., n\}$ we get

$$\sum_{i=0}^{n-1} |F(x_{i+1}) - F(x_i)| \;\leq\; \int |f| dm \,.$$

The absolute continuity follows from

$$\lim_{m(A) \to 0} \int_A |f| dm \;=\; 0 \,.$$

(see exam). ■

Lemma 15.2. *Let $f : [a, b] \to \mathbb{R}$ be a of bounded variation. Then f is the difference of two monotone functions f_1, f_2. If in addition f is absolutely continuous, then f_1 and f_2 may be assume absolutely continuous.*

$$\|f\|_{BV} \;=\; f_1(b) + f_2(b) - f(a) \,.$$

Proof. For any partition π we define

$$p(f, \pi) = \sum_{i=0}^{n-1} \max\{f(x_{i+1}) - f(x_i), 0\}$$

$$n(f,\pi) = \sum_{i=0}^{n-1} \max\{-f(x_{i+1}) + f(x_i), 0\}$$
$$t(f,\pi) = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \ .$$

Then

$$t(f,\pi) \ = \ p(f,\pi) + n(f,\pi)$$

and

$$f(b) - f(a) \ = \ \sum_{i=0}^{n} (f(x_{i+1}) - f(x_i)) \ = \ p(f,\pi) - n(f,\pi) \ .$$

This implies

$$f(b) - f(a) + n(f,\pi) \ = \ p(f,\pi) \ .$$

Now, we take the supremum over all partitions and still have

$$f(b) - f(a) + \sup_\pi n(f,\pi) \ = \ \sup_\pi p(f,\pi) \ .$$

Moreover,

$$\sup_\pi t(f,\pi) = \sup_\pi [p(f,\pi) + n(f,\pi)] \ = \ \sup_\pi [2p(f,\pi) - (f(b) - f(a))]$$
$$= \sup_\pi p(f,\pi) + \sup_\pi p(f,\pi) - (f(b) - f(a)) \ = \ \sup_\pi p(f,\pi) + \sup_\pi n(f,\pi) \ .$$

For $a \le x \le b$ we define

$$g(x) \ = \ \sup_{\pi = \{a = x_0 < \ldots < x_n = x\}} p(f,\pi)$$

and

$$h(x) \ = \ \sup_{\pi = \{a = x_0 < \ldots < x_n = x\}} n(f,\pi) \ .$$

Then g and h are increasing functions and

$$f(x) - f(a) + h(x) \ = \ g(x) \ .$$

This yields

$$f(x) \ = \ g(x) - h(x) + f(a) \ .$$

Moreover,

$$\|f\|_{BV} \ = \ g(b) - f(a) + h(b) \ .$$

If in addition f is absolute continuous then it follows by the definition that $\sum_i |x_{i+1} - x_i| < \delta$ implies

$$\sup_{i \in J} \sum_{i \in J} |g(x_{i+1}) - g(x_i)| \ \le \ \sup_{\sum_j |y_{j+1} - y_j| < \delta} \sum_j \max\{f(y_{j+1}) - f(y_j), 0\} \ .$$

Thus we can work with same relation between ε and δ for g and h. ■

THEOREM 15.3. *Let $F : [a,b] \to \mathbb{R}$ be an absolute continuous function of bounded variation. Then there exists a function $f \in L_1[a,b]$ such that*

$$F(t) \;=\; F(a) + \int_a^t f(s)ds$$

and $\|f\|_1 \;=\; \|F\|_{BV}$.

PROOF. We may assume $F(a) = 0$. Let $F \;=\; F_1 - F_2$ such that F_1 and F_2 are positive

$$F_1(b) + F_2(b) \;=\; \|F\|_{BV}$$

and such that F_1, F_2 are absolutely continuous. We define the measure on $A_{\mathbb{R}}$.

$$\nu((s,t]) \;=\; F_1(t) - F(s) \;.$$

Using the absolute continuity it is not hard to check that

$$\nu((s,t]) \;=\; \sum_j \nu((s_j,t_j])$$

for every disjoint decomposition. Thus ν extends to a σ additive measure on the borel sets which is absolutely continuous with respect to the Lebesque measure. Thus ν extends to Lebesgue measurable set. By the Radon-Nikodym theorem we find a measurable function f_1 such that

$$F_1((s,t]) \;=\; \int_s^t f_1 dm \;.$$

Then

$$\int f_1 dm \;=\; F_1(b) - F_1(a) \;=\; F_1(b) \;.$$

We apply the same argument to F_2 and find a positive element f_2 such that

$$\int f_2 dm \;=\; F_2(b) \;.$$

Thus $f = f_1 - f_2$ satisfies the assertion by Lemma 15.1. ■

16. Connection to the Riemann integral

DEFINITION 16.1. *Let f be bounded. The* oscillation *of f at x is defined by*

$$\omega(x) = \lim_{\delta \to 0} \sup\{|f(z) - f(y)| : |z - x| < \delta, |y - x| < \delta\}$$

REMARK 16.2. *$\omega(x)$ well-defined. In particular, $\omega(x) \leq 2 \sup |f| < \infty$.*

REMARK 16.3. *f continuous at $x \Leftrightarrow \omega(x) = 0$*

REMARK 16.4. *$m(\{x : \omega(x) \neq 0\}) = \lim_{n\to\infty} m(\{x : \omega(x) > 1/n\})$*

DEFINITION 16.5. *(Darboux Integral)*
For a partition $\pi =< a = x_0, x_1, \ldots, x_n = b >$ we define the upper and the lower sum by

$$\overline{S}(\pi, f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f$$

$$\underline{S}(\pi, f) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f$$

A bounded function f is called Darboux-integrable if $\forall \epsilon > 0$ $\exists$ partition π such that

$$\overline{S}(\pi, f) - \underline{S}(\pi, f) < \epsilon$$

REMARK 16.6. *A function is Darboux-integrable if and only if it is Riemann-integrable (see Royden for a precise definition).*

PROPOSITION 16.7. *A bounded function is Riemann integrable on $[a, b]$ if and only if the set of discontinuity points has measure 0.*

PROOF. "$\Longrightarrow$": Let f be Darboux-integrable. Let $\gamma > 0$ and $\delta > 0$. We define $\epsilon = \gamma\delta$. By definition there exists a partition $\pi =< 0 = x_0, x_1, \ldots, x_n = 1 >$ such that

$$\overline{S}(\pi, f) - \underline{S}(\pi, f) < \epsilon.$$

Consider $x \in (x_i, x_{i+1})$ such that $\omega(x) \geq \gamma$. Then

$$\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \geq \gamma.$$

Observe that points with 'large' (and hence certain discontinuity) points satisfy

$$\{x : \omega(x) \geq \gamma\} = \bigcup_{i, \sup f - \inf f \geq \gamma} (x_i, x_{i+1}) \cup \{x_0, \ldots, x_n\}$$

Hence,

$$
\begin{aligned}
m(\{x : \omega(x) \geq \gamma\}) &\leq \sum_{\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f \geq \gamma} |x_{i+1} - x_i| \\
&\leq \frac{1}{\gamma} \sum |x_{i+1} - x_i| (\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f) \\
&\leq \frac{1}{\gamma} \big(\overline{S}(\pi, f) - \underline{S}(\pi, f) \big) \\
&\leq \frac{\epsilon}{\gamma} = \delta
\end{aligned}
$$

Since δ is arbitrary, we get $m(\{x : \omega(x) \geq \gamma\}) = 0$. However, $\gamma > 0$ is arbitrary and thus Remark (16.4) implies $m(\{x : \omega(x) > 0\}) = 0$. This means the set of discontinuity points has measure 0.

"$\Longleftarrow$" Define $\omega_\delta(x) = \sup\{|f(z) - f(y)| : |z - x| < \delta, |y - x| < \delta\}$.

By assumption we have $m(\{x : \omega(x) > 0\}) = 0$. This implies

$$m(\{x : \omega(x) \geq \gamma\}) = 0$$

for all $\gamma > 0$. Therefore, we deduce from the monotonicity of $\omega_\delta(x)$ that

(16.1)

$$m(\{x : \omega(x) \geq \gamma\}) = m(\{x : \lim_{\delta \to 0} \omega_\delta(x) \geq \gamma\}) \ = \ \lim_{\delta \to 0} m(\{x : \omega_\delta(x) \geq \gamma\}) = 0$$

Let $\gamma > 0$ and $\epsilon = \frac{\gamma}{2 \sup |f|}$. By (16.1) we deduce the existence of some k such that

$$m(\{x : \omega_{1/k}(x) \geq \gamma\}) < \epsilon \, .$$

Choose $m > k$ and $\pi =< x_0, x_1, \ldots, x_m >=< 0, \frac{1}{m}, \frac{2}{m}, \ldots, \frac{m}{m} >$.

Define

$$S \ = \ \{j \in \{1, \ldots, m\} : \exists_{x \in [x_{j-1}, x_j]} \ \ \omega_{1/k}(x) \geq \gamma\} \, .$$

Then we get

$$\sum_{j \in S} (\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f)(x_{j-1} - x_j) \leq \gamma \sum_{j \in S} |x_{j+1} - x_j| \ \leq \ \gamma(b - a) \, .$$

If $j \notin S$, then

$$[x_{j-1} - x_j] \subset \{x : \omega_{1/k}(x) > \gamma\} \, .$$

This implies

$$
\begin{aligned}
&\sum_{j \notin S} (\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f)(x_{j-1} - x_j) \leq 2 \sup_{[0,1]} |f| \, m(\bigcup_{j \notin S} [x_{j-1}, x_j]) \\
&\leq \ 2 \sup_{[a,b]} |f| \, m(\{x : \omega_{1/k}(x) > \gamma\}) \ \leq \ 2 \sup_{[a,b]} |f| \epsilon \ \leq \ \gamma \, .
\end{aligned}
$$

Putting the pieces together, we find

$$\sum_j (\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f)(x_{j-1} - x_j) \ \leq \ \gamma(b-a) + \gamma \ = \ \gamma(1+b-a) \ .$$

Since $\gamma > 0$ is arbitrary we deduce that f is Darboux integrable. ∎

17. A covering lemma and the local maximal function

Among the most important tools in analysis are the results known as covering theorems. For the real line, the covering sets are intervals. In this section, we present a simple form of the best covering theorem for the real line. The original result is an extension by J. Aldaz ["A general covering lemma for the real line", *Real Anal. Exchange* **17**(1991/92), 394–398] of a lemma of T. Radó ("Sur un problème relatif à un théorème de Vitali", *Fundamenta Mathematicae* **11**(1928), 228–229.) In the Aldaz result, the constant 3 is improved to $2+\varepsilon$ for an arbitrary $\varepsilon > 0$, and the result is valid for any finite Borel measure, not just Lebesgue measure m on a bounded interval.

THEOREM 17.1 (Rado-Aldaz). *Given an arbitrary collection $\mathcal{I}$ of non-degenerate intervals, all contained in a fixed bounded interval J, the set $\cup_{I\in\mathcal{I}} I$ is measurable, and there is a finite disjoint subset $\{I_1, \cdots, I_n\} \subseteq \mathcal{I}$ such that*

$$m(\cup_{I\in\mathcal{I}} I) \leq 3 \cdot \sum_{k=1}^{n} m(I_k).$$

PROOF. Let L be the set of left end points which are not in $\bigcup_{I\in\mathcal{I}} I$. Let $y \neq z \in L$ and $I_y = [y, b)$ or $I_y = [y, b]$, and $I_z = [z, c)$ or $I_z = [y, c]$. We may also assume that $y < z$. If $b > z$, then z is an interior point. Hence we must have $b \leq z$. This shows that

$$L \cap (y, b) \ = \ \emptyset$$

According to our assumption the intervals I are not degenerate, i.e. have non-empty interior. Thus for every $y \in L$ we can find a rational number $r(y) \in (y, b)$ such that $y \neq z$ implies $r(y) \neq r(z)$. Then the map $r : L \to \mathbb{Q}$ is injective and hence L is at most countable. The same arguments applies for the right end points. For this part of the argument we need the fact that the intervals are not degenerated to a point. Therefore, $(\cup_{I\in\mathcal{I}} I) \setminus (\cup_{I\in\mathcal{I}} I^\circ)$ is at most a countable set. By Lindelöf's theorem (i.e., any union of open intervals equals the union of a countable subcollection) we may assume that $\mathcal{I}$ itself is a countable collection $\{I_n\}$, whence $\cup_{I\in\mathcal{I}} I = \cup_{n=1}^{\infty} I_n$ is measurable.

Now since all the intervals are contained in the bounded interval J,

$$m(\cup_{n=1}^{\infty} I_n) = \lim_{N\in\mathbb{N}} m(\cup_{n=1}^{N} I_n) < +\infty.$$

We employ Rado's result after first choosing N so that

$$\frac{3}{2} \cdot m(\cup_{n=1}^{N} I_n) \geq m(\cup_{n=1}^{\infty} I_n) = m(\cup_{I\in\mathcal{I}} I).$$

Ordering the finite collection, we discard the first interval if it is covered by the remaining intervals. Otherwise, we keep the first interval and consider the second. In either case, this does not change the measure $m(\cup_{n=1}^{N} I)$. Continuing in this way, we may assume that each I_n in our finite collection contains a point x not in any other interval of the collection. Now, we order these points and reorder the corresponding intervals so that for any indices i, j, and k with $i < j < k$ we have $x_i < x_j < x_k$ and thus $I_i \subseteq (-\infty, x_j)$ and $I_k \subseteq (x_j, +\infty)$. That is, the points are given the ordering inherited from $\mathbb{R}$, and the intervals are given the same ordering as their points. Since the intervals with even indices form a disjoint collection, as do the intervals with odd indices, the desired subset of $\mathcal{I}$ is whichever of these two families has the greater total measure. For example, if we choose the even indices, then

$$3 \cdot m(\cup I_{2n}) = \frac{3}{2} \cdot 2 \cdot m(\cup I_{2n}) \geq \frac{3}{2} \cdot m(\cup_{n=1}^{N} I_n) \geq m(\cup_{I \in \mathcal{I}} I). \qquad \blacksquare$$

The next result is refined maximal inequality due to Jürgen Bliedtner and P. Loeb ("Limit Theorems via Local Maximal Functions", preprint.) Here, we let f be a nonnegative integrable function on $\mathbb{R}$. We set $\mathcal{I}(x, r)$ equal to the set of intervals I containing x with positive length $m(I) \leq r$, and we set

$$M(f, r, x) := \sup_{I \in \mathcal{I}(x,r)} \frac{1}{m(I)} \int_I f \ dm.$$

Since $M(f, r, x)$ decreases as r decreases, we may set

$$M(f, x) := \lim_{r \to 0+} M(f, r, x),$$

where the limit is understood to be $+\infty$ if $M(f, r, x) = +\infty$ for all $r > 0$.

PROPOSITION 17.2. *Let E be a bounded subset of $\mathbb{R}$. Fix $\alpha > 0$, and let $E_\alpha = \{x \in E : M(f, x) > \alpha\}$. Then the outer measure of E_α satisfies*

$$m^*(E_\alpha) \leq \frac{3}{\alpha} \cdot \int_{\mathbb{R}} f \ dm.$$

PROOF. Given $x \in E_\alpha$, there is an interval $I_x \in \mathcal{I}(x, 1)$ such that

$$\alpha \cdot m(I_x) \leq \int_{I_x} f \, dm.$$

These intervals form a collection $\mathcal{I}$ that cover E_α, so by Theorem 17.1, there is a finite disjoint subcollection $\{I_1, \cdots, I_n\} \subset \mathcal{I}$ such that

$$m^*(E_\alpha) \leq m(\cup_{I \in \mathcal{I}} I) \leq 3 \cdot \sum_{k=1}^{n} m(I_k) \leq \frac{3}{\alpha} \sum_{k=1}^{n} \int_{I_k} f \, dm \leq \frac{3}{\alpha} \cdot \int_{\mathbb{R}} f \ dm. \qquad \blacksquare$$

Remark 17.3. *Let*

$$\tilde{M}_n(f)(x) = \sup_{I \in I(x,n)} \frac{1}{m(I)} \int_I f(x) dm(x) .$$

The argument above still shows that

$$m^*(\{x : \tilde{M}_n(f)(x) > \alpha\} \leq \frac{3}{\alpha} \int_{\mathbb{R}} f dm$$

holds for every positive function and every n. One can then send n to ∞.

Proposition 17.4. *Let E be a bounded measurable subset of $\mathbb{R}$, and let f be a nonnegative integrable function that vanishes almost everywhere on E. Then $M(f, x) = 0$ for almost all $x \in E$.*

Proof. Given $\alpha > 0$ and an $\varepsilon > 0$, we may fix an open set $U \supseteq E$ so that $\int_U f\, dm < \varepsilon\alpha/3$. Now

$$E_\alpha := \{x \in E : M(f, x) > \alpha\} = \{x \in E : M(f \cdot \chi_U, x) > \alpha\}.$$

Therefore, $m^*(E_\alpha) \leq \frac{3}{\alpha} \int_U f \ dm < \varepsilon$. Since ε is arbitrary, $m^*(E_\alpha) = 0$, and the result follows. ■

18. The Lebesgue Differentiation Theorem

THEOREM 18.1 (Lebesgue Differentiation). *Let f be an integrable function on $\mathbb{R}$. Then each of the following equalities holds almost everywhere on $\mathbb{R}$.*

$$\begin{aligned}
\lim_{r\to 0+} \frac{1}{2r}\int_{[x-r,x+r]} f\,dm &= f(x),\\
\lim_{r\to 0+} \frac{1}{r}\int_{[x,x+r]} f\,dm &= f(x),\\
\lim_{r\to 0+} \frac{1}{r}\int_{[x-r,x]} f\,dm &= f(x).
\end{aligned}$$

PROOF. We need only prove the result for $f \geq 0$ and x in a fixed, bounded, open interval J. By Lusin's Theorem, for any $\varepsilon > 0$, there is a compact set $K \subset J$ with $m(J \setminus K) < \varepsilon$ such that $f|_K$ is continuous on K. Let $O = \mathbb{R} \setminus K$ be the complement of K and $a = \inf_K$, $b = \sup_K$. We may write $O = \bigcup_j I_j$ as a countable union of open intervals. We may assume $I_1 = (-\infty, a)$. There we define $h(x) = f(a)\frac{|a|^2}{|x|^2}$. On $I_2 = (b, \infty)$ we define $h(x) = f(b)\frac{|b|^2}{|x|^2}$. All the other components are bounded. Now we replace inductively every I_j by the maximal interval $I_j' \supset I_j$ such that $I_j' \cap K = \emptyset$. On such an I_j' we may use a piecewise linear function. In this way we construct a continuous function h such that $h|_K = f|_K$. Note that $g = h - 1_K f$ is measurable and integrable by construction. Moreover, g vanishes on K. By Proposition 17.4 all the limits for g vanish on K. Thus we see that

$$f = h - g + f1_{\mathbb{R}\setminus K}$$

holds on $\mathbb{R}$. By the continuity of h, each of the limit results holds almost everywhere on K for h, g and $f \cdot 1_{\mathbb{R}\setminus K}$, and thus for f. Since ε is arbitrary, the limit result is established for almost all points of J. ■

We have already established that if f is Lebesgue integrable on $[a, b]$, then $F(x) := \int_a^x f\,dm = \int_{[a,x]} f\,dm$ is continuous. We used the fact that $\forall\varepsilon > 0$, $\exists\delta > 0$ such that if $mA < \delta$, then $\int_A |f| < \varepsilon$. It was important, however, that points have measure 0. For differentiablility, we have the following results that generalizes part of the Fundamental Theorem of Calculus. It uses the fact that we are integrating with respect to Lebesgue measure, so that the length of an interval is its measure.

THEOREM 18.2. *Suppose f is Lebesgue integrable on $[a, b]$, and $F(x) = \int_a^x f\,dm + C$ where C is a constant. Then $F'(x) = f(x)$ for almost all $x \in [a, b]$.*

Proof. For $\Delta x = r > 0$ and $x + \Delta x \leq b$,

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{r} \int_{[x,x+r]} f \, dm.$$

For $\Delta x = -r < 0$ and $x + \Delta x \geq a$,

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{-r} \cdot -\int_{[x-r,x]} f \, dm = \frac{1}{r} \int_{[x-r,x]} f \, dm.$$

The result now follows from the previous theorem. ■

Since $\int_a^a f = 0$, we can think of the constant C as $F(a)$. The theorem says, we can differentiate the indefinite integral of a Lebesgue integrable function and get back the integrand almost everywhere. We would like to integrate the derivative of any reasonable function and get back the function we started with. We know this is impossible with the Cantor function since the Cantor function's derivative is 0 almost everywhere.

19. Lusin's Theorem and applications

Egoroff's Theorem says that on a set of finite measure, almost everywhere convergence of measurable functions to a finite limit is uniform convergence off of a set of small measure. A consequence (see Problem 31 on Page 74) is Lusin's Theorem, which says that on a set of finite measure, any finite measurable function f can be restricted to a compact set K of almost full measure to form a continuous function. We will present a new simple proof of Lusin's theorem due to Erik Talvila and P. Loeb. Let us recall the following result proved in a homework:

LEMMA 19.1. *Given a measurable set $A \subseteq \mathbb{R}$ with $m(A) < +\infty$, and given $\varepsilon > 0$, there is a compact set $K \subseteq A$ with $m(A \setminus K) < \varepsilon$.*

PROOF. We already know that there is a closed subset F of A with $m(A \setminus F) < \varepsilon/2$. Since the sequence

$$F \cap [-n, n] \nearrow F,$$

and $m(F) < +\infty$, there is an n_0 such that $m(F \setminus [-n_0, n_0]) < \varepsilon/2$. The desired compact set is $F \cap [-n_0, n_0]$. $\square$

THEOREM 19.2 (Lusin). *Fix a measurable set $A \subseteq \mathbb{R}$ with $m(A) < +\infty$, and let f be a real-valued measurable function with domain A. For any $\varepsilon > 0$, there is a compact set $K \subseteq A$ with $m(A \setminus K) < \varepsilon$ such that the restriction of f to K is continuous.*

PROOF. Let $\langle V_n \rangle$ be an enumeration of the open intervals with rational endpoints in $\mathbb{R}$. Fix compact sets $K_n \subseteq f^{-1}[V_n]$ and $K'_n \subseteq A \setminus f^{-1}[V_n]$ for each n so that $m(A \setminus (K_n \cup K'_n)) < \varepsilon/2^n$. Now, for $K := \bigcap_n (K_n \cup K'_n)$, $m(A \setminus K) < \varepsilon$. Given $x \in K$ and an n with $f(x) \in V_n$, $x \in O := \widetilde{K'_n}$ and $f[O \cap K] \subseteq V_n$. $\blacksquare$

This result is true in quite general settings. In the general setting, you may see this result stated just for Borel measurable functions. The domain of f should have the property that sets of finite measure can be approximated from the inside by compact sets, and for the range, there should be a countable collection of open sets V_n such that for each open set O and each $y \in O$ there is an n with $y \in V_n \subseteq O$. (This is called the Second Axiom of Countability.)

COROLLARY 19.3. *Let A be a measurable set such that $m(A) < \infty$. Let $f : A \to \mathbb{R}$ be measurable function and $\varepsilon > 0$. Then there exists a step function $h : \mathbb{R} \to \mathbb{R}$ such that*

$$m(|f - h| \geq \varepsilon) < \varepsilon\,.$$

Moreover, if f is bounded than

$$\sup |h| \ \leq \ \sup |f| \ .$$

Proof. Let K be such that $f|K$ is continuous and

$$m(A\mathcal{K}) < \varepsilon \ .$$

Being compact we known that K is bounded , say $K \subset [-N, N]$. Since $f_{|}K$ is continuous it is also uniform continuous. Thus we may find $0 < \delta < \varepsilon$ such that

$$t, s \in K \text{and} |t - s| < \delta \Longrightarrow |f(t) - f(s)| < \varepsilon \ .$$

Let $n > \delta^{-1}$ and $x_i = -N + \frac{i}{n}$, $i = 0, ..., 2Nn$. Let S be the collection of indices such that there exists $i \in K$ such that $[x_i, x_{i+1}) \cap K \neq \emptyset$. For such $i \in S$ we may pick $y_i \in [x_i, x_{i+1}]$. We define the step function

$$h \ = \ \sum_{i \in S} f(y_i) 1_{[x_i, x_{i+1})} \ .$$

Let $s \in K$. Choose $i = 0, ..., 2Nm$ such that $x_i \leq s < x_{i+1}$. Then $K \cap [x_i, x_{i+1}) \cap K \neq \emptyset$ and $|y_i - s| < \frac{1}{n} < \delta$. We get

$$|h(s) - f(s)| \ = \ |f(y_i) - f(s)| < \varepsilon \ .$$

Thus

$$m(|h - f| \geq \varepsilon) \ \leq \ m(A \setminus K) < \varepsilon \ .$$

Since h is constructed using the elements $f(y_i)$ we also get

$$\sup_{x \in \mathbb{R}} |h(x)| \ \leq \ \sup_{x \in K} |f(x)| \ .$$

This implies the second assertion. ■

Corollary 19.4. *Let $A \subset \mathbb{R}$ be a measurable set and $f : A \to \mathbb{R}$ be a measurable function and $\varepsilon > 0$. Then there exists a continuous function h such that*

$$m(|f - h| > \varepsilon) < \varepsilon \ .$$

Moreover, we can choose h such that

$$\sup |h| \ \leq \ \sup |f| pl.$$

Proof. It suffices to show that for every simple function $f = \sum_{i=1}^m r_i 1_{[x_i,x_{i+1})}$ we can find a continuous h with

$$\mu(|f-h| > \varepsilon) < \varepsilon \quad \text{and} \quad |h| \le |f| \, .$$

It is easily shown by induction that

$$\mu(|(\sum_i f_i) - (\sum_i h_i)| > \sum_i \varepsilon_i) \ \le \ \sum_i \mu(|f_i - h_i| > \varepsilon_i) \, .$$

Therefore it suffices to consider $f_i = 1_{[x_i,x_{i+1})}$. Let $0 < 2\delta < x_{i+1} - x_i$ we define

$$h_{i,\delta}(t) \ = \ \begin{cases} \delta^{-1}(t - x_i) & \text{if } x_i < t \le x_i + \delta \, , \\ 1 & \text{if } x_i + \delta \le t \le x_{i+1} - \delta \, , \\ \delta^{-1}(x_{i+1} - t) & \text{if } x_{i+1} - \delta \le t \le x_{i+1} \\ 0 & \text{else} \end{cases} \, .$$

Note that $h_{i,\delta} \le 1_{[x_i,x_{i+1})}$ is continuous and that

$$m(|h_{i,\delta} - 1_{[x_i,x_{i+1})}| > 0) < 2\delta \, .$$

Let δ such that $\frac{2\delta}{m} < \min_i(x_{i+1} - x_i)$. Then we may define

$$h \ = \ \sum_i r_i h_{i,\frac{\delta}{m}}$$

Then we have

$$m(|f-h| > \delta) \ \le \ \sum_{i=1}^m m(r_i|1_{[x_i,x_{i+1})} - h_{i,\frac{\delta}{m}}| > \frac{\delta}{m}) < 2m\frac{\delta}{m} < 2\delta \, .$$

For the second assertion, we note that

$$|h| \le |f| \, .$$

Therefore we also control the sup-norm. ∎

20. Convex Functions

In the following theorem, the book uses $[0,1]$ with Lebesgue measure for the probability space.

THEOREM 20.1 (Jensen's Inequality). *Let $(X, \mathcal{A}, \mu)$ be a probability space (i.e., $\mu(X) = 1$), and fix an integrable function f on X with range contained in an open interval I. Let φ be a convex function on I. Then*

$$\varphi(\int_X f \; d\mu) \le \int_X \varphi \circ f \; d\mu.$$

Note : It may be that $\varphi \circ f$ is not integrable; in this case, the integral on the right equals $+\infty$.

PROOF. Let $y = \int_X f \; d\mu$. Then $y \in I$ since y is a "general average" of the values of f. Another way to see this is to note that if a is the left endpoint of I and a is finite, then $a = a \cdot \mu(X) \le \int_X f \; d\mu$, and equality means that f takes the value a almost everywhere, which it does not. A similar proof holds for a finite right endpoint of I. Now, if

$$\beta = \sup_{\substack{x<y \\ x \in I}} \frac{\varphi(y) - \varphi(x)}{y - x}$$

then for any $z > y$ in I,

$$\beta \le \frac{\varphi(z) - \varphi(y)}{z - y}$$

It follows that for $z \ge y$ in I,

$$*) \quad \varphi(z) \ge \varphi(y) + \beta \cdot (z - y).$$

If $z < y$ in I, then by the definition of β,

$$\begin{aligned} \beta &\ge \frac{\varphi(y) - \varphi(z)}{y - z} = \frac{\varphi(z) - \varphi(y)}{z - y}, \text{ so} \\ \varphi(z) - \varphi(y) &\ge (z - y)\beta. \end{aligned}$$

This means that (*) holds for all z in I. For any $t \in X$, we may let $z = f(t)$, and we get

$$\varphi(f(t)) \ge \varphi(y) + \beta(f(t) - y).$$

Recall φ is continuous so $\varphi \circ f$ is measurable. The inequality means that $\varphi(f(t))$ is bounded below by an integrable function. Integrating both sides of the inequality,we get

$$\int_X \varphi \circ f \; d\mu \ge \varphi(\int_X f \; d\mu) + \beta \cdot 0. \qquad \blacksquare$$

EXAMPLE 20.2. A primary example is given by setting $\varphi(x) = e^x$ on $I = \mathbb{R}$. If X is a set consisting of n points with each point having probability $1/n$, then letting x_i be the image under f of the n^{th} point, we have

$$\exp\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{e^{x_1} + \cdots + e^{x_n}}{n}.$$

Putting $y_i = e^{x_i}$, this gives the classical inequality between the arithmetic and geometric mean for positive numbers:

$$(y_1 y_2 \cdots y_n)^{\frac{1}{n}} \leq \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

A generalization works with positive weights α_i such that $\Sigma\alpha_i = 1$. This gives

$$y_1^{\alpha_1} \cdots y_n^{\alpha_n} \leq \alpha_1 y_1 + \cdots + \alpha_n y_n.$$

For a countable number of points, we have

$$\Pi_{n=1}^{\infty} y_i^{\alpha_i} \leq \Sigma_{n=1}^{\infty} \alpha_i y_i.$$

Infinite products are covered in the graduate complex variables course.