Practice problems

For the style of questions I ask, look at the homework problems.

(1) Let (X, d) be a metric space and $f : X \to \mathbb{R}$ be continuous. Let (Ω, Σ, μ) be a measure space and $g : \Omega \to X$ such that

$$g^{-1}(O) \in \Sigma$$

holds for every open set O. Show that $f \circ g$ is measurable (with respect to Σ). Assume that only

$$g^{-1}(B(x_0, r)) \in \Sigma$$

holds for every open ball $B(x_0, r) = \{x : d(x, x_0) < r\}$. Show that if there exists a countable dense set in X, then $f \circ g$ is still measurable. Can you find a counterexample for $X = \mathbb{R}$ assuming that f is not continuous?

(2) Problem 28 on page 71.

(3) Problem 17 on page 94.

(4) Let f_n be defined by

$$f_n(x) \ = \ 1_{[n,n+1]} \sin(n^2 x) \ .$$

Show that $\sum_n f_n$ is not in $\mathcal{L}^1$. However,

$$\lim_{N \to \infty} \int_{\mathbb{R}} \sum_{n=1}^{N} f_n dm$$

exists.

(5) Problem 25 on page 96.