CHAPTER 1

Metric Spaces

1. Definition and examples

Metric spaces generalize and clarify the notion of distance in the real line. The definitions will provide us with a useful tool for more general applications of the notion of distance:

DEFINITION 1.1. *A metric space is given by a set* X *and a distance function* $d : X \times X \to \mathbb{R}$ *such that*

i) *(Positivity) For all* $x, y \in X$

$$0 \leq d(x,y) \; .$$

ii) *(Non-degenerated) For all* $x, y \in X$

$$0 \; = \; d(x,y) \quad \Leftrightarrow \quad x \; = \; y \; .$$

iii) *(Symmetry) For all* $x, y \in X$

$$d(x,y) \; = \; d(y,x)$$

iv) *(Triangle inequality) For all* $x, y, z \in X$

$$d(x,y) \; \leq \; d(x,z) + d(z,y) \; .$$

Examples:

i) $X = \mathbb{R}$, $d(x,y) = |x-y|$.

ii) $X = \mathbb{R}^2 \; = \; \mathbb{R} \times \mathbb{R}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_1(x,y) \; = \; |x_1 - y_1| + |x_2 - y_2| \; .$$

iii) $X = \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$d_2(x,y) \; = \; \left(|x_1 - y_1|^2 + |x_2 - y_2|^2 \right)^{\frac{1}{2}} \; .$$

iv) Let $X = \{p_1, p_2, p_3\}$ and

$$d(p_1, p_2) \; = \; d(p_2, p_1) \; = \; 1 \; ,$$

$$d(p_1, p_3) \; = \; d(p_3, p_1) \; = \; 2 \; ,$$

$$d(p_2,p_3) \;=\; d(p_3,p_2) \;=\; 3\,.$$

Can you find a triangle (p_1,p_2,p_3) in the plane with these distances?

v) Let $X=\{p_1,p_2,p_3\}$ and

$$\begin{aligned} d(p_1,p_2) &= d(p_2,p_1) \;=\; 1\,, \\ d(p_1,p_3) &= d(p_3,p_1) \;=\; 2\,, \\ d(p_2,p_3) &= d(p_3,p_2) \;=\; 4\,. \end{aligned}$$

Can you find a triangle (p_1,p_2,p_3) in the plane with these distances?

vi) The French railway metric (Chicago suburb metric) on $X=\mathbb{R}^2$ is defined as follows: Let $x_0=(0,0)$ be the origin, then

$$d_{SNCF}(x,y) \;=\; \begin{cases} d_2(x,y) & \text{if there exists a } t\in\mathbb{R} \text{ such that } x_1=ty_1 \\ & \text{and } x_2=ty_2 \\ & \\ d_2(x,x_0)+d_2(x_0,y) & \text{else} \end{cases}\,.$$

Exercise: Show that the railroad metric satisfies the triangle inequality.

It is by no means trivial to show that d_2 satisfies the triangle inequality. In the following we write $0=(0,...,0)$ for the origin in $\mathbb{R}^n$.

LEMMA 1.2. *Let $x,y\in\mathbb{R}^n$, then*

$$|\sum_{i=1}^n x_iy_i| \;\le\; \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n |y_i|^2\right)^{\frac{1}{2}}$$

LEMMA 1.3. *On $\mathbb{R}^n$ the metric*

$$d_2(x,y) \;=\; \left(\sum_{i=1}^n |x_i-y_i|^2\right)^{\frac{1}{2}}$$

satisfies the triangle inequality.

PROOF. Let $x,y,z\in\mathbb{R}^n$. Then we deduce from Lemma 1.2

$$\begin{aligned} d(x,y)^2 &= \sum_{i=1}^n |x_i-y_i|^2 \;=\; \sum_{i=1}^n |(x_i-z_i)-(y_i-z_i)|^2 \\ &= \sum_{i=1}^n |(x_i-z_i)|^2 - 2\sum_{i=1}^n (x_i-z_i)(y_i-z_i) + \sum_{i=1}^n |y_i-z_i|^2 \\ &\le d(x,z)^2 + 2d(x,y)d(y,z) + d(y,z) \end{aligned}$$

$$= \quad (d(x,z)+d(y,z))^2 \, .$$

Hence,

$$d(x,y) \ \le \ d(x,z)+d(y,z)$$

and the assertion is proved. ■

More examples:

(1) Let n be a prime number. On $\mathbb{Z}$ we define

$$dd_n(x,y) \ = \ n^{-\max\{m\in\mathbb{N}\,:\, m \text{ divides x-y}\}} \, .$$

The n-adic metric satisfies a stronger triangle inequality

$$dd_n(x,y) \ \le \ \max\{dd_n(x,z), dd_n(z,y)\} \, .$$

(2) Let $1 \le p < \infty$. Then

$$d_p(x,y) \ = \ \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}$$

defines a metric n $\mathbb{R}^n$.

(3) For $p = \infty$

$$d_\infty(x,y) \ = \ \max_{i=1,..,n} |x_i - y_i|$$

also defines a metric on $\mathbb{R}^n$.

Project 1: Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Show Minkowski's inequality.

$$xy \le \frac{x^p}{p} + \frac{y^q}{q} \tag{1.1}$$

holds for all $x, y > 0$. **Hint:** the function $f(x) = -\ln x$ is convex on $(0,\infty)$.

Proof of the triangle inequality for d_p. The triangle inequality for $p = 1$ is obvious. We will fist show

$$|\sum_{i=1}^n x_i y_i| \ \le \ \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}} \tag{1.2}$$

whenever $\frac{1}{p} + \frac{1}{q} = 1$. Let $t > 0$. We first observe that

$$\begin{aligned} |\sum_{i=1}^n x_i y_i| = \sum_{i=1}^n |tx_i||t^{-1}y_i| \ &\le \ \sum_{i=1}^n \frac{1}{p}|tx_i|^p + \frac{1}{q}|t^{-1}y_i|^q \\ &= \frac{t^p}{p} \sum_{i=1}^n |x_i|^p + \frac{t^{-q}}{q} \sum_{i=1}^n |y_i|^q \, . \end{aligned}$$

What is best choice of t? Make

$$t^p \sum_{i=1}^n |x_i|^p \;=\; t^{-q} \sum_{i=1}^n |y_i|^q$$

i.e.

$$t^{p+q} \;=\; \frac{\sum_{i=1}^n |y_i|^q}{\sum_{i=1}^n |x_i|^p} \,.$$

This yields

$$\begin{aligned} |\sum_{i=1}^n x_i y_i| \le t^p \sum_{i=1}^n |x_i|^p \;&=\; \frac{\left(\sum_{i=1}^n |y_i|^q\right)^{\frac{p}{p+q}}}{\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{p}{p+q}}} \sum_{i=1}^n |x_i|^p \\ &= \left(\sum_{i=1}^n |y_i|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n |x_i|^p\right)^{1-\frac{1}{q}} \end{aligned}$$

Now, we proof the triangle inequality. Let $x = (x_i)$, (y_i) and $z = (z_i)$ in $\mathbb{R}^d$. Then we apply (1.2)

$$\begin{aligned} d_p(x,y)^p = \sum_{i=1}^d |x_i - y_i|^p \;&\le\; \sum_{i=1}^d |x_i - y_i|^{p-1}(|x_i - z_i| + |z_i - y_i|) \\ &\le \sum_{i=1}^d |x_i - y_i|^{p-1}|x_i - z_i| + \sum_{i=1}^d |x_i - y_i|^{p-1}|z_i - y_i| \\ &\le \left(\sum_{i=1}^d (|x_i - y_i|^{p-1})^q\right)^{\frac{1}{q}} \left(\left(\sum_{i=1}^d |z_i - x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^d |z_i - y_i|^p\right)^{\frac{1}{p}}\right) \,. \end{aligned}$$

However, $1 = 1/p + 1/q$ implies $p - 1 = p/q$ and thus $q(p-1) = p$. Hence we get

$$d_p(x,y)^p \;\le\; d_p(x,y)^{p-1}(d_p(x,z) + d_p(z,y)) \,.$$

If $x \neq y$ we may divide and deduce the assertion. ■

2. Excursion: Convex functions

DEFINITION 2.1. *Let I be an interval. A function $f : I \to \mathbb{R}$ is called convex if*

$$f(\lambda x + (1-\lambda)y) \ \le \ \lambda f(x) + (1-\lambda) f(y)$$

holds for all $x, y \in I$, $0 < \lambda < 1$.

LEMMA 2.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous, differnetiable on (a,b) such that f' is increasing. Then f is convex.*

PROOF. Let $x \in [a,b]$. We will show that

$$g(z) \ = \ \frac{f(y+z) - f(y)}{z}$$

is monotone increasing on $(0, b-x)$. Indeed, by the fundamental theorem and change of variables we deduce for $z_1 < z_2$ and $\lambda \ = \ \frac{z_1}{z_2}$ $(s = \lambda t$, $ds = \lambda dt$

$$\begin{aligned} g(z_1) = \int_0^{z_1} f'(s) \frac{ds}{z_1} \ &= \ \int_0^{z_2} f'(\lambda t) \frac{\lambda dt}{z_1} \ = \ \int_0^{z_2} f'(\lambda t) \frac{dt}{z_2} \\ &\le \int_0^{z_2} f'(t) \frac{dt}{z_2} \ = \ g(z_2) \, . \end{aligned}$$

Now, we fix $y < x$ and $u = \lambda x + (1-\lambda) y = y + \lambda(x-y)$, $z_1 = \lambda(x-y)$, $z_2 = x - y$. Then, we get

$$\frac{f(y+z) - f(y)}{\lambda(x-y)} \ \le \ \frac{f(x) - f(y)}{(x-y)} \, .$$

This implies

$$f(\lambda x + (1-\lambda) y) \le f(y) + \lambda(f(x) - f(y)) \ = \ \lambda f(x) + (1-\lambda) f(y) \, . \qquad \blacksquare$$

PROOF OF 1.1. Let $x, y > 0$. Since $-\ln x$ is convex we have

$$-\ln(\frac{1}{p}x^p + \frac{1}{q}y^q) \ \le \ \frac{1}{p}(-\ln x^p) + \frac{1}{q}(-\ln y^q) \, .$$

This shows by the monotonicity of exp that

$$\frac{1}{p}x^p + \frac{1}{q}y^q \ \ge \ e^{\ln x + \ln y} \ = \ xy \, .$$

Minkowski's inequality is proved. $\blacksquare$

3. Continuous functions between metric spaces

Continuous functions 'preserve' properties of metric spaces and allow to describe deformation of one metric space into another. There are three different (but equivalent) ways of defining continuity, the ε-δ-criterion, the sequence criterion and the topological criterion. Each of them is interesting in its own right.

Definition 3.1. *Let* (X, d) *and* (Y, d') *be metric spaces. A map* $f : X \to Y$ *is called continuous if for every* $x \in X$ *and* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$d(x, y) < \delta \quad \Longrightarrow \quad d'(f(x), f(y)) < \varepsilon \, . \tag{3.1}$$

Let us use the notation

$$B(x, \delta) \;=\; \{y \,:\, d(x, y) < \delta\} \, .$$

For a subset $A \subset X$, we also use the notation

$$f(A) \;=\; \{f(x) \,:\, x \in A\} \, .$$

Similarly, for $B \subset Y$

$$f^{-1}(B) \;=\; \{x \in X \,:\, f(x) \in B\} \, .$$

Then (3.1) means

$$f(B(x, \delta)) \;\subset B(f(x), \varepsilon) \, .$$

Or in a very non-formal way

$$\text{f maps small balls into small balls} \; .$$

Our aim is to prove a criterion for continuity in terms of so called open sets. This criterion illustrates simultaneously the role of open sets and its interaction with continuity and has a genuinely geometric flavor.

Definition 3.2. *A subset* O *of a metric space is called* $\underline{\textit{open}}$ *if*

$$\forall x \in O \;: \exists \delta > 0 \;: B(x, \delta) \subset O \, .$$

Examples:

$$O \;=\; (-1, 1) \, , \; O \;=\; \mathbb{R} \, , \; O \;=\; (-1, 1) \times (-2, 2)$$

are open in $\mathbb{R}$, $(\mathbb{R}^2, d_2)$ respectively.

Remark 3.3. *The sets* $B(x, \varepsilon)$, $x \in X$, $\varepsilon > 0$ *are open.*

PROPOSITION 3.4. *Let* (X, d), (Y, d') *be metric spaces and* $f : X \to Y$ *be a map.* f *is contiuous iff* $f^{-1}(O)$ *is open for all open subsets* $O \subset Y$.

PROOF. $\Rightarrow$: We assume that f is continuous and O is open. Let $x \in f^{-1}(O)$, i.e. $f(x) \in O$. Since O is open, there exists an $\varepsilon > 0$ such that $B(f(x), \varepsilon) \subset O$. By continuity, there exists a $\delta > 0$ such that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon) \subset O\,.$$

Therefore

$$B(x, \delta) \subset f^{-1}(O)\,.$$

Since $x \in f^{-1}(O)$ was arbitrary, we deduce that $f^{-1}(O)$ is open.
$\Leftarrow$: Let $x \in X$ and $\varepsilon > 0$. Let us show that

$$B(f(x), \varepsilon)$$

is a on open subset of (Y, d'). Indeed, let $y \in B(f(x), \varepsilon)$ define $\varepsilon' = \varepsilon - d'(y, f(x))$. Let $z \in Y$ such that

$$d(z, y) < \varepsilon'$$

then

$$d(f(x), z) \;\leq\; d(f(x), y) + d(y, z) \;<\; d(f(x), y) + \varepsilon - d'(y, f(x)) \;=\; \varepsilon\,.$$

Thus

$$B(y, \varepsilon - d'(f(x), y)) \;\subset B(f(x), \varepsilon)\,.$$

By the assumption, we see that $f^{-1}(B(f(x), \varepsilon))$ is an open set. Since $x \in f^{-1}(B(f(x), \varepsilon))$, we can find a $\delta > 0$ such that

$$B(x, \delta) \subset f^{-1}(B(f(x), \varepsilon))\,.$$

Hence, for all $\tilde{x}$ with $d(x, \tilde{x}) < \delta$, we have

$$d'(f(x), f(\tilde{x})) < \varepsilon\,.$$

The assertion is proved. ■

Examples:

(1) Let (X, d) be a metric space and $x_0 \in X$ be a point , then $f(x) = d(x, x_0)$ is continuous. Indeed, the triangle inequality implies

$$d(d(x, x_0), d(d(y, 0)) \;=\; |d(x, x_0) - d(y, x_0)| \;\leq\; d(x, y)$$

This easily implies the assertion.

(2) On $\mathbb{R}^n$ with the standard euclidean metric $d = d_2$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ defined by $f(x) = d(x, 0)x$ is continuous.
(3) (Exercise) The function $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x) = (\cos(x_1), \sin(x_2), \cos(x_1))$ is continuous.

Definition 3.5. *Let (X, d), (Y, d') be a metric space. The space $C(X, Y)$ is the set of all continuous functions from X to Y. Let $x_0 \in X$ be a point. Then*

$$C_b(X, Y) \;=\; \{f : X \to Y \;:\; f \text{ is continuous and } \sup_{x\in X} d'(f(x), f(x_0)) < \infty\}$$

is the subset of bounded continuous functions.

Proposition 3.6. *Let (X, d), (Y, d') be metric spaces and $x_0 \in X$. Then $C_b(X, Y)$ equipped with*

$$d(f, g) \;=\; \sup_{x\in X} d'(f(x), g(x))$$

is a metric space.

Problem: Show that d is not well-defined on $C(\mathbb{R}, \mathbb{R})$.

Proof: $d(f, g) = 0$ if and only if $f(x) = g(x)$ for all $x \in X$. This means $f = g$. Let us show that d is well-defined. Indeed, if $f, g \in C_b(X, Y)$. Then

$$\begin{aligned}\sup_x d'(f(x), g(x)) &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + d'(g(x_0), g(x)) \\ &\le \sup_x d'(f(x), f(x_0)) + d'(f(x_0), g(x_0)) + \sup_x d(g(x_0), g(x))\end{aligned}$$

is finite. Let h be a third function and $x \in X$. Than

$$d'(f(x), g(x)) \;\le\; d'(f(x), h(x)) + d(h(x), g(x)) \;\le\; d(f, h) + d(h, g)\ .$$

Taking the supremum yields the assertion. ■

Proposition 3.7. *Let (X, d) be a metric space. Then $C(X, \mathbb{R})$ is closed under (pointwise-) sums, products and multiplication with real numbers. ($C(X, \mathbb{R})$ is an algebra over $\mathbb{R}$).*

Remark 3.8. *Let $X = \mathbb{N}$ and $d(x, y) = 1$ of $x \ne y$ and $d(x, y) = 0$ for $x = y$. (This is called the discrete metric). Then $C(X, \mathbb{R})$ is an infinite dimensional vector space.*

Proof of 3.7. Let $f, g \in C(X, \mathbb{R})$ be continuous and $x \in X$. Consider $x' \in X$. Then

$$fg(x) - fg(y) \;=\; f(x)g(x) - f(y)g(y) \;=\; (f(x) - f(y))g(x) + f(y)(g(x) - g(y))$$

$$= (f(x) - f(y))g(x) + f(x)(g(x) - g(y)) + (f(y) - f(x))(g(x) - g(y)) .$$

Let $\varepsilon > 0$ and $\tilde{\varepsilon} = \min\{\varepsilon, 1\}$. We may choose $\delta_1 > 0$ such that

$$d(f(x), f(y))(1 + |g(x)|) < \frac{\tilde{\varepsilon}}{3}$$

holds for all $d(x, y) < \delta_1$. Similarly, we may choose $\delta_2 > 0$ such that

$$d(g(x), g(y))(1 + |f(x)|) < \frac{\tilde{\varepsilon}}{3} .$$

Let $\delta = \min(\delta_1, \delta_2)$ and $d(x, y) < \delta$. Then we deduce that

$$d(fg(x), fg(y)) = |fg(x) - fg(y)| < \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}}{3} + \frac{\tilde{\varepsilon}^2}{9} < \tilde{\varepsilon} \le \varepsilon .$$

Thus fg is again continuous. The other assertions are easier. ■

Corollary 3.9. *The polynomials on $\mathbb{R}$ are continuous.*

Lemma 3.10. *Let $1 \le p \le \infty$ and $x, y \in \mathbb{R}^n$, then*

$$\frac{1}{n^{\frac{1}{p}}} d_p(x, y) \le d_\infty(x, y) \le d_p(x, y) .$$

Proof. The last inequality is obvious. For the first one, we consider $x, y \in \mathbb{R}^n$ and $1 \le p < \infty$, then by estimating every element in the sum against the maximum

$$d_p(x, y)^p = \sum_{i=1}^{n} |x_i - y_i|^p \le n \max\{|x_i - y_i|^p\} .$$

Taking the p-th root, we deduce the assertion. ■

Corollary 3.11. *Let $1 \le p, q \le \infty$, then the identity map $id : (\mathbb{R}^n, d_p) \to (\mathbb{R}^n, d_q)$ is continuous.*

Proof. We have for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$

$$B_{d_p}(x, \frac{\varepsilon}{n}) \subset B_{d_q}(x, \varepsilon) .$$

This easily implies the assertion. ■

Corollary 3.12. *The metrics d_p define the same open sets on $\mathbb{R}^n$.*

Definition 3.13. *Let (X, d) be a metric space. We say that a sequence (x_n) converges to x_0 if for all $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ we have*

$$d(x_n, x_0) < \varepsilon \, .$$

In this case we write

$$\lim_n x_n \; = \; x$$

or more explicitly

$$d - \lim_n x_n \; = \; x \, .$$

A sequence (x_n) is convergent, if there exists $x \in X$ with $\lim_n x_n = x$.

Examples: $d_2 - \lim_n \frac{1}{n} \; = \; 0$, $dd_3 - \lim_n 3^n \; = \; 0$. (What axioms of the natural numbers are involved?).

Proposition 3.14. *Let (X, d), (Y, d') be metric spaces and $f : X \to Y$ be a map. Then f is continuous if for every convergent sequence (x_n) in X*

$$\lim_n f(x_n) \; = \; f(\lim_n x_n) \, .$$

Proof: $\Rightarrow$: Let $x = \lim_n x_n$ and $\varepsilon > 0$, then there exists a $\delta > 0$ such that

$$d(y, x) < \delta \Rightarrow d'(f(y), f(x)) < \varepsilon \, .$$

Let $n_0 \in \mathbb{N}$ be such that

$$d(x_n, x) < \delta$$

for all $n > n_0$, then

$$d'(f(x_n), f(x)) < \varepsilon$$

for all $n > n_0$. Hence

$$\lim_n f(x_n) \; = \; f(x) \, .$$

$\Leftarrow$ Let $x \in X$ and assume in the contrary that

$$\exists \varepsilon > 0 \, \forall \delta > 0 \exists y \; : d(y, x) < \delta \text{ and } d'(f(x), f(y)) \geq \varepsilon \, .$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence (x_k) such that

$$d(x_k, x) \; < \; \frac{1}{k} \quad \text{and} \quad d'(f(x_k), f(x)) \geq \varepsilon' \, .$$

and thus

$$\lim_k x_k \; = \; x \, .$$

By assumption, we have

$$\lim_k f(x_k) \; = \; f(x) \, .$$

Hence, there exists a k_0 such that for all $k > k_0$

$$d(f(x_k), f(x)) < \varepsilon \, .$$

a contradiction. ■

4. Complete metric spaces and completion

Complete metric space are crucial in understanding existence of solutions to many equations. Complete spaces are also important in understanding spaces of integrable functions. We will review basic properties here and show the existence of a completion.

We will say that a sequence in a metric space is a Cauchy sequence of for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$.

Definition 4.1. *A metric space (X, d) is called complete, if every Cauchy sequence converges.*

Proposition 4.2. *The space $(\mathbb{R}^2, d_1)$ is complete.*

Proof: Let x_n be a Cauchy sequence in $(\mathbb{R}^2, d_1)$. Then $x_n = (x_n(1), x_n(2))$ is a sequence of pairs.

Claim: The sequences $(x_n(1))_{n\in\mathbb{N}}$ and $(x_n(2))_{n\in\mathbb{N}}$ are Cauchy sequences.

Indeed, let $\varepsilon > 0$, then there exists an n_0 such that

$$d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$. In particular, we have

$$|x_n(1) - x_m(1)| \;\le\; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \;\le\; d_1(x_n, x_m) < \varepsilon$$

for all $n, m > n_0$ and

$$|x_n(2) - x_m(2)| \;\le\; |x_n(1) - x_m(1)| + |x_n(2) - x_m(2)| \;\le\; d_1(x_n, x_m) < \varepsilon \, .$$

Therefore, $(x_n(1))$ and $(x_n(2))$ are Cauchy.

Since $\mathbb{R}$ is complete, we can find $x(1)$ and $x(2)$ such that

$$\lim_n x_n(1) \;=\; x(1) \quad \text{and} \quad \lim_n x_n(2) \;=\; x(2) \, .$$

Claim: $\lim_n x_n \;=\; (x(1), x(2))$.

Indeed, Let $\varepsilon > 0$ and choose n_1 such that

$$|x_n(1) - x(1)| < \frac{\varepsilon}{2}$$

for all $n > n_1$. Choose n_2 such that

$$|x_n(2) - x(2)| < \frac{\varepsilon}{2}$$

for all $n > n_2$. Set $n_0 = \max\{n_1, n_2\}$, then for every $n > n_0$, we have

$$d_1(x_n, (x(1), x(2)) = |x_n(1) - x(1)| + |x_n(2) - x(2)| < \varepsilon$$

Thus

$$\lim_n x_n = x$$

and the assertion is proved. ■

Examples:

(1) Let $X = \mathbb{R} \setminus \{0\}$ and $d(x,y) = |x-y|$, them (X,d) is not complete. The sequences $(\frac{1}{n})$ is Cauchy and does not converge.

(2) Let p be a prime number. On the set of integers, we define

$$dd_p(z,w) = p^{-n} ,$$

where $n = \max\{n : p^n \text{ divides } (z-w)\}$. This satisfies the triangle inequality. The sequence (x_n) given by $x_n = p + p^2 + \cdots + p^n$ is a non convergent Cauchy sequence.

THEOREM 4.3. *Let $n \in \mathbb{N}$. The space $(\mathbb{R}^n, d_2)$ is a complete metric space.*

PROOF. Similar as in Proposition 4.2 using the following Lemma . ■

LEMMA 4.4. *Let $x, y \in \mathbb{R}^n$, then*

$$d_2(x,y) \le \sum_{i=1}^{n} |x_i - y_i| .$$

PROOF. We proof this by induction on $n \in \mathbb{N}$. The case $n = 1$ is obvious. Assume the assertion is true for n and let $x, y \in \mathbb{R}^{n+1}$. We define the element $z = (x_1, ..., x_n, y_{n+1})$, then we deduce from the triangle inequality

$$\begin{aligned} d_2(x,y) &\le d_2(x,z) + d_2(z,y) \\ &= \left(\sum_{i=1}^{n+1} |x_i - z_i|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n+1} |z_i - y_i|^2\right)^{\frac{1}{2}} \\ &= |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} . \end{aligned}$$

To apply the induction hypothesis, we define $\tilde{x} = (x_1, ..., x_n)$ and $\tilde{y} = (y_1, .., y_n)$. Then the induction hypothesis yields

$$\left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} = d_2(\tilde{x}, \tilde{y}) \leq \sum_{i=1}^{n} |x_i - y_i| .$$

Hence,

$$\begin{aligned} d_2(x, y) &\leq |x_{n+1} - y_{n+1}| + \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} \\ &\leq |x_n - y_n| + \sum_{i=1}^{n} |x_i - y_i| \\ &= \sum_{i=1}^{n+1} |x_i - y_i| . \end{aligned}$$

The assertion is proved. ∎

Definition 4.5. *A subset $C \subset X$ is called closed if $X \setminus C$ is open.*

Proposition 4.6. *Let C be closed subset of a complete metric space (X, d), then $(C, d|_{C\times C})$ is complete.*

Proof. Let $(x_n) \subset C$ be Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$x = \lim_n x_n .$$

We have to show $x \in C$. Assume $x \notin C$. Then there exists a $\delta > 0$ such that $B(x, \delta) \subset X \setminus C$. By definition of the limit there exists n_0 such that $d(x_n, x) < \delta$ for all $n > n_0$. Set $n = n_0 + 1$. Then $d(x_n, x) < \delta$ implies $x_n \in X \setminus C$ and $x_n \in C$ by definition. This contradiction finished the proof. ∎

Theorem 4.7. *Let (Y, d') be complete metric space. Let $h \in C(X, Y)$ and*

$$C_h(X, Y) = \{f \in C(X, Y) : \sup_{x\in X} d'(f(x), h(x)) < \infty\}$$

Then $C_g(X, Y)$ is complete with respect to

$$d(f, g) = \sup_{x\in X} d'(f(x), g(x)) .$$

PROOF. Let $(f_n) \subset C_h(X,Y)$ be Cauchy sequence. This means that for every $\varepsilon > 0$ there exists an n_0 such that

$$\sup_{x\in X} d'(f_n(x), f_m(x)) < \frac{\varepsilon}{2}\,. \tag{4.1}$$

In particular, for fixed $x \in X$, $f_n(x)$ is Cauchy. Therefore $f(x) := \lim_m f_m(x)$ is a well-defined element in Y. We fix $n > n_0$ and consider $m \geq n_0$ such that

$$d'(f_m(x), f(x)) \;\leq\; \frac{\varepsilon}{3}\,.$$

This implies

$$d'(f_n(x), f(x)) \;\leq\; d'(f_n(x), f_m(x)) + d'(f_m(x), f(x)) \;\leq\; \frac{5}{6}\varepsilon$$

for all $x \in X$. In particular,

$$\sup_{n\geq n_0}\sup_{x\in X} d'(f_n(x), f(x)) \;\leq\; \frac{5}{6}\varepsilon\,. \tag{4.2}$$

Let us show that f is continuous. Let $z \in X$ and $\varepsilon > 0$. Choose n_0 according to (4.1). Choose $n = n_0 + 1$. Let $\delta > 0$ such that $d(x,y) < \delta$ implies $d'(f_n(x), f_n(y)) < \varepsilon$. Then, we have

$$d'(f(x), f(y)) \;\leq\; d'(f(x), f_n(x)) + d'(f_n(x), f_n(y)) + d'(f_n(y), f(y)) < 3\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, we see that f is continuous. Moreover, (4.2) implies that f_n converges to f. Finally, (4.2) for $\varepsilon = 1$ implies that

$$\sup_x d(f(x), h(x)) \;\leq\; \sup_x d(f(x), f_n(x)) + \sup_x d(f_n(x), h(x)) < \infty$$

implies that $f \in C_h(X,Y)$. ■

DEFINITION 4.8. *Let (X,d) be a metric space and $C \subset X$. $O \subset X$ is called sense if for ever $x \in C$ and $\varepsilon > 0$ $B(x,\varepsilon) \cap O \neq \emptyset$.*

DEFINITION 4.9. *Let $O \subset X$ be a subset. Then*

$$\bar{O} \;=\; \cap_{O\subset C, C\, closed} C$$

is called the closure.

LEMMA 4.10. *O is dense in $\bar{O}$ and $\bar{O}$ is closed.*

Proof. Let $x \in \bar{O}$. Assume $B(x,\varepsilon) \cap O = \emptyset$. Then $C = X \setminus B(x,\varepsilon)$ contains O. Thus

$$\bar{O} \subset C\,.$$

This implies that $x \notin \bar{O}$, a contradiction. Now, we show that $\bar{O}$ is closed. Indeed, let $y \notin \bar{O}$. Then there has to be a closed set C such that $O \subset C$ but $y \notin C$. This means $y \in X \setminus C$ which is open. Hence there exists $\delta > 0$ such that

$$B(y,\delta) \subset X \setminus C$$

By definition every element $z \in B(y,\delta)$ does not belong to $\bar{O}$. This means $B(y,\delta) \subset X \setminus \bar{O}$. ■

Theorem 4.11. *Let (X,d) be a non-empty metric space. For every $x \in X$ we define*

$$f_x(y) \;=\; d(x,y)\,.$$

Let $x_0 \in X$. The map $f : X \to C_{f_{x_0}}(X,\mathbb{R})$ satisfies the following properties.

i) *$d(f(x), d(f(y)) \;=\; d(x,y)$,*
(1) *The closure $C = \overline{f(X)}$ is complete,*
(2) *$f(X)$ is dense in the closure $C = \overline{f(X)}$.*

Proof. Let $x,y \in X$ and $z \in X$. Then the 'converse triangle' ineqquality implies

$$|f_x(z) - f_y(z)| \;=\; |d(x,z) - d(y,z)| \;\le\; d(x,y)\,.$$

Moreover,

$$|f_x(z) - f_x(y)| \;=\; |d(x,z) - d(x,y)| \;\le\; d(z,y)\,.$$

Therefore $f_x \in C_{f_{x_0}}(X,\mathbb{R})$ for every $x \in X$ and

$$d(f_x, f_y) \;\le\; d(x,y)\,.$$

However,

$$d(f_x,f_y) \;\ge\; |f_x(x) - f_y(x)| \;=\; |0 - d(y,x)| \;=\; d(y,x)\,.$$

This shows *i*). According to Proposition 4.6 and Theorem 4.7, we see that C is complete. According to Lemma 4.10, we deduce that $f(X)$ is dense in C. ■

Project: On $C([0,1])$ we define

$$d_1(f,g) \;=\; \int |f(s) - g(s)|ds\,.$$

Show that $(C([0,1]), d_1)$ is not complete.

Project: In the literature you can find another description of the completion of a metric space. Find it and describe it.

5. Unique extension of densely defined uniformly continuous functions

In this section we will show that the completion C constructed in Theorem 4.11 is unique (in some sense). This is based on a simple observation-the unique extension. This principle is very often used in analysis.

DEFINITION 5.1. *Let (X,d), (Y,d') be metric spaces. A function $f : X \to Y$ is called uniform continuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$d(x,y) < \delta \quad \Rightarrow \quad d'(f(x),f(y)) < \varepsilon \, .$$

PROPOSITION 5.2. *Let $O \subset C$ be a dense set and $f : O \to Y$ be uniformly continuous function with values in a complete metric space. Then there exists a unique continuous function $\tilde{f} : O \to Y$ such that $\tilde{f}(x) = f(x)$ for all $x \in O$.*

PROOF. Let $x \in X$. Since $B(x, \frac{1}{n}) \cap O$ is not empty, we may find $(x_n) \subset O$ such that $\lim_n x_n = x$. We try to define

$$\tilde{f}(x) \;=\; \lim_n f(x_n) \, .$$

Let us show that this is well-defined. So we consider another Cauchy sequence (x'_n) such that $\lim_n x'_n = x$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$d'(f(x',y)) < \varepsilon$$

holds for $d(x',y) < \delta$. We may find n_0 such that

$$d(x_n,x) < \frac{\delta}{2}$$

and

$$d(x'_n,x) < \frac{\delta}{2}$$

holds for all $n,n' > n_0$. Thus

$$d'(f(x'_n),f(x_n)) < \varepsilon \, .$$

This argument also shows that $(f(x_n))$ is Cauchy and hence $\tilde{f}(x)$ is well-defined. If $x \in O$, we may choose for (x_n) the constant sequence $x_n = x$ and hence $\tilde{f}(x) = f(x)$. Now, we want to show that $\tilde{f}$ is uniformly continuous. Indeed, let $\varepsilon > 0$, then there exists $\delta > 0$ such that $d(x',y') < \delta$ implies

$$d(f(x'),f(y')) < \frac{\varepsilon}{2} \, .$$

Given $x, y \in C$ with $d(x,y) < \delta$, we may find (x_n) converging to x and (y_n) converging to y such that

$$d(x_n, x) < \frac{\delta - d(x,y)}{2} \, .$$

Thus for all $n \in \mathbb{N}$ we have

$$d(x_n, y_n) \ \le \ d(x,y) + d(x_n, x) + d(y_n, y) < \delta \, .$$

This implies

$$d(f(x), f(y)) \ = \ \lim_n d(f(x_n), f(y_n)) < \frac{\varepsilon}{2} \, .$$

This shows that $\tilde{f}$ is uniformly continuous. If g is another continuous function such that $g(x) = f(x)$ holds for elements $x \in O$, then we may choose a Cauchy sequence (x_n) converging to x and get

$$g(x) = \lim_n g(x_n) \ = \ \lim_n f(x_n) \ = \ f(x) \, . \qquad \blacksquare$$

Example If $f : (0,1] \to \mathbb{R}$ is uniformly continuous, then f is bounded (why). $f(x) = 1/x$ is not uniformly continuous.

THEOREM 5.3. *The completion of a metric space is unique. More precisely, let* C *be the set constructed in Theorem 4.11. Let* C' *be a complete metric space and* $\iota' : X \to C'$ *be uniformly continuous with uniformly continuous inverse* ${\iota'}^{-1} : \iota(X) \to X$ *such that* $\iota'(X)$ *is dense. Then there is a bijective, bicontinuous map* $u : C \to C'$ *such that* $u(\iota(x)) = \iota'(x)$.

PROOF. The map $\iota' \iota^{-1} : \iota(X) \to C'$ is uniformly continuous and hence admits a unique continuous extension $u : C \to C'$. Also $\iota {\iota'}^{-1} : \iota'(X) \to C$ admits a unique extension $v : C' \to C$. Note that $vu : C \to C$ is an extension of the map $vu(\iota(x)) = \iota(x)$. Thus there is only one extension, namely the identity. This show $vu = id$. Similarly $uv = id$. Thus $v = u^{-1}$ and u is bijective and bi-continuous. $\blacksquare$

Project: Find the completion of $(\mathbb{Z}, d_3)$.

6. Closed and Compact Sets

Let (X, d) be a metric space. We will say that a subset $A \subset X$ is *closed* if $X \setminus A$ is open.

Proposition 6.1. *Let (X, d) be a complete metric space and $C \subset X$ a subset. C is closed iff every Cauchy sequence in C converges to an element in C.*

Proof: Let us assume C is closed and that (x_n) is a Cauchy sequence with elements in C. Let $x = \lim_n x_n$ be te limit and assume $x \notin C$. Since $X \setminus C$ is open

$$B(x, \varepsilon) \subset X \setminus C$$

for some $\varepsilon > 0$. Then there exists an n_0 such that $d(x_n, x) < \varepsilon$ for $n > n_0$. In particular,

$$x_{n_0+1} \in B(x, \varepsilon)$$

and thus $x_{n_0+1} \notin C$, a contradiction.

Now, we assume that every Cauchy sequence with values in C converges to an element in C. If $X \setminus C$ is not open, then there exists an $x \notin C$ and no $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset X \setminus C \,.$$

I.e. for every $n \in \mathbb{N}$, we can find $x_n \in C$ such that

$$d(x, x_n) < \frac{1}{n} \,.$$

Hence, $\lim x_n = x \in C$ but $x \notin C$, contradiction. ■

The most important notion in this class is the notion of compact sets. We will say that a subset $C \subset X$ is *compact* if For every collection (O_i) of open sets such that

$$C \subset \bigcup_i O_i \;=\; \{x \in X \mid \exists_{i \in I} x \in O_i\}$$

There exists $n \in \mathbb{N}$ and $i_1, ..., i_n$ such that

$$C \subset O_{i_1} \cup \cdots \cup O_{i_n} \,.$$

In other words

Every open cover of C has a finite subcover .

DEFINITION 6.2. *Let $X \subset \bigcup O_i$ be an open cover. Then we say that (V_j) is an open subcover if*

$$X \subset \bigcup_j V_j$$

all the V_j are open and for every j there exists an i such that

$$V_j \subset O_i \, .$$

It is impossible to explain the importance of 'compactness' right away. But we can say that there would be no discipline 'Analysis' without compactness. The most clarifying idea is contained in the following example.

PROPOSITION 6.3. *The set $[0,1] \subset \mathbb{R}$ is compact.*

PROPOSITION 6.4. *Let $B \subset X$ be closed set and $C \subset X$ be a compact set, then*

$$B \cap C$$

is compact

Proof: Let $B \cap C \subset \bigcup O_i$ be an open cover. then

$$C \subset (X \setminus B) \cup \bigcup_i O_i$$

is an open cover for C, hence we can find a finite subcover

$$C \subset (X \setminus B) \cup O_{i_1} \cup \cdots \cup O_{i_n} \, .$$

Thus

$$B \cap C \ \subset O_{i_1} \cup \cdots \cup O_{i_n}$$

is a finite subcover. ■

LEMMA 6.5. *Let (X,d) be a metric space and $D \subset X$ be a countable dense set in X, then for every subset $C \subset X$ and every open cover*

$$C \subset \bigcup_i O_i$$

we can find a countable subcover of balls.

Proof: Let us enumerate D as $D = \{d_n \,|\, n \in \mathbb{N}\}$. Let $x \in C$ and find $i \in I$ and $\varepsilon > 0$ such that

$$x \in B(x, \varepsilon) \subset O_i \,.$$

Let $k > \frac{2}{\varepsilon}$. By density, we can find an $n \in \mathbb{N}$ such that

$$d(x, d_n) < \frac{1}{2k} \,.$$

Then

$$x \in B(d_n, \frac{1}{2k}) \subset B(x, \frac{1}{k}) \subset B(x, \varepsilon) \subset O_i \,.$$

Let us define

$$M \;=\; \{(n,k) \mid \exists_{i \in I} B(d_n, \frac{1}{2k}) \subset O_i\} \,.$$

Then $M \subset \mathbb{N}^2$ is countable and hence there exists a map $\phi : \mathbb{N} \to M$ which is surjective (=onto). Hence for $V_m = B(d_{\phi_1(m)}, \frac{1}{2\phi_2(m)})$, ϕ_1, ϕ_2 the 2 components of ϕ we have

$$C \subset \bigcup_m V_m$$

and (V_m) is a countable subcover of balls of the original cover (O_i). ■

Theorem 6.6. *Let (X, d) be a metric space. Let $C \subset X$ be a subset. Then the following are equivalent*

- i) a) *Every Cauchy sequence of elements in C converges to a limit in C.*
  - b) *For every $\varepsilon > 0$ there exists points $x_1, ..., x_n \in X$ such that*

$$C \subset B(x_1, \varepsilon) \cup \cdots \cup B(x, \varepsilon) \,.$$

- ii) *Every sequence in C has a convergent subsequence.*
- iii) *C is compact.*

Proof: *i)* $\Rightarrow$ *ii)*. Let (x_n) be a sequence. Inductively, we will construct infinite subset $A_1 \supset A_2 \supset A_3 \cdots$ and y_1, y_2, y_3,... in X such that

$$\forall_{l \in A_j} \;:\; d(x_l, y_j) < 2^{-j-1} \,.$$

Put $A_0 = \mathbb{N}$. Let us assume $A_1 \supset A_2 \supset \cdots A_n$ and $y_1, ..., y_n$ have been constructed. We put $\varepsilon = 2^{-n-2}$ and apply condition *i)b)* to find $z_1, ..., z_m$ such that

$$C \;\subset\; B(z_1, \varepsilon) \cup \cdots \cup B(z_m, \varepsilon) \,.$$

We claim that there must be a $1 \leq k \leq m$ such that

$$A_n(k) \;=\; \{l \in A_n \,|\, x_n \in B(z_k, \varepsilon)\}$$

has infinitely many elements. Indeed, we have

$$A_n(1) \cup \cdots \cup A_n(m) \ = \ A_n \ .$$

If they were all finite, then a finite union of finite sets would have finitely many elements. However A_n is infinite. Contradiction! Thus, we can find a k with $A_n(k)$ infinite and put $A_{n+1} = A_n(k)$ and $y_{n+1} = z_k$. So the inductive procedure is finished. Now, we can find an increasing sequence (n_j) such that $n_j \in A_j$ and deduce

$$d(x_{n_j}, x_{n_{j+1}}) \ \leq \ d(x_{n_j}, y_j) + d(y_j, x_{n_{j+1}}) < \frac{1}{2}2^{-j} + \frac{1}{2}2^{-j} \ = \ 2^{-j}$$

because $n_j \in A_j$ and $n_{j+1} \in A_{j+1} \subset A_j$. Thus (x_{n_j}) is Cauchy. Indeed, be induction, we deduce for $j < m$ that

$$\begin{aligned} d(x_{n_j}, x_{n_m}) \ &\leq \ d(x_{n_j}, x_{n_{j+1}}) + d(x_{n_{j+1}}, x_{n_{j+2}}) \cdots d(x_{n_{m-1}}, x_{n_m}) \\ &\leq \ 2^{-j} \sum_{k=0}^{m-1} 2^{-k} \ = \ 2^{1-j} \ . \end{aligned}$$

This easily implies the Cauchy sequence condition. By a) it converges to some $x \in C$. We got our convergent subsequence.
ii) $\Rightarrow$ *iii)*: We will first show *ii)* $\Rightarrow$ *i)b)*. Indeed, let $\varepsilon > 0$ and assume for all $n \in \mathbb{N}$, $y_1, ..., y_n \in C$ we may find

$$x(n, y_1, ..., y_n) \in C \setminus (B(y_1, \varepsilon) \cup \cdots B(y_n, \varepsilon)) \ .$$

Then we define $x_1 \in C$ and find $x_2 \in C \setminus B(x_1, \varepsilon)$. Then we find $x_3 \in C \setminus B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$. Thus inductively we find $x_n \in C$ such that

$$d(x_n, x_k) \geq \varepsilon$$

for all $1 \leq k \leq n$. It is easily seen that (x_n) has no convergent subsequence. Thus *i)b)* is showed (with points in C). For every $\varepsilon_k = \frac{1}{k}$ we find these points $y_1^k, ..., y_{m(k)}^k \in C$ such that

$$C \subset B(y_1^k, \frac{1}{k}) \cup \cdots \cup B(y^{m(k)}, \frac{1}{k}) \ .$$

Then, we see that $D = \{y_j^k : k \in \mathbb{N}, 1 \leq j \leq m(k)\}$ is dense in C. Therefore, we may work with the closure $\tilde{X} = \bar{D}$ and show that C is compact in $\tilde{X}$. (It will then be automatically compact in X). By Lemma 6.5, we may assume that

$$C \subset \bigcup_k O_k$$

and O_k's open. If we can find an n such that

$$C \subset O_1 \cup \cdots \cup O_n$$

the assertion is proved. Assume that is not the case and choose for every $n \in \mathbb{N}$ an $x_n \in C \setminus O_1 \cup \cdots \cup O_n$. According to the assumption, we have a convergent subsequence, i.e. $\lim_k x_{n_k} = x \in C$. Then $x \in O_{n_0}$ for some n_0 and there exists a $\varepsilon > 0$ such that

$$B(x,\varepsilon) \ \subset O_{n_0} \, .$$

By convergence, we find a k_0 such that $d(x,x_{n_k}) < \varepsilon$ for all $k > k_0$. In particular, we find a $k > k_0$ such that $n_k > n_0$. Thus

$$x_{n_k} \in B(x,\varepsilon) \in O_{n_0} \subset O_1 \cup \cdots \cup O_{n_k} \, .$$

Contradicting the choice of the (x_n)'s. We are done.
iii) $\Rightarrow$ *i)b)* Let $\varepsilon > 0$ and then

$$C \ \subset \bigcup_{x \in C} B(x,\varepsilon) \, .$$

thus a finite subcover yields *b)*.
iii) $\Rightarrow$ *i)a)* Let (x_n) be a Cauchy sequence. Assume it is not converging to some element $x \in C$. This means

$$\forall x \in C \exists \varepsilon(x) > 0 \forall n_0 \exists n > n_0 \; d(x_n,x) > \varepsilon \, . \tag{6.1}$$

Then

$$C \subset \bigcup_{x \in C} B(x, \frac{\varepsilon(x)}{2}) \, .$$

Let

$$C \subset B(y_1, \frac{\varepsilon(y_1)}{2}) \cup \cdots \cup B(y_1, \frac{\varepsilon(y_1)}{2})$$

be a finite subcover (compactness). Then there exists at least one $1 \le k \le m$ such that

$$A_k \ = \ \{n \in \mathbb{N} \mid d(x_n,y_k) < \frac{\varepsilon(y_k)}{2}\}$$

is infinite. Fix that k and apply the Cauchy criterion to find n_0 such that

$$d(x_n,x_{n'}) < \frac{\varepsilon(y_k)}{2}$$

for all $n,n' > n_0$. By (6.1), we can find an $n > n_0$ such that

$$d(x_n,y_k) > \varepsilon(y_k) \, .$$

Since A_k is infinite, we can find an $n' > n_0$ in A_k thus

$$\begin{aligned} \varepsilon(y_k) \ &< \ d(x_{n'},y_k) \ \le \ d(x_n,x_{n'}) + d(x_{n'},y_k) \\ &< \ \frac{\varepsilon(y_k)}{2} + \frac{\varepsilon(y_k)}{2} \ = \ \varepsilon(y_k) \, . \end{aligned}$$

A contradiction. Thus the Cauchy sequence has to converge to some point in C. ■

Corollary 6.7. *Every intervall* $[a,b] \subset \mathbb{R}$ *with* $a < b \in \mathbb{R}$ *is compact*

Proof: It is easy to see that $X \setminus [a,b]$ is open. Hence, by Proposition 6.1 $[a,b]$ is complete, i.e. i)a) is satisfied. Given $\varepsilon > 0$, we can find $k > \frac{1}{\varepsilon}$. For $m > k(b-a)$ we derive

$$[a,b] \subset \bigcup_{j=0}^{m} B(a + \frac{j}{k}, \varepsilon) \ .$$

Thus the Theorem 6.6 applies. ■

Lemma 6.8. *Let* $r > 0$ *and* $n \in \mathbb{N}$, *the set* $C_r = [-r,r]^n$ *is compact.*

Proof: Let $x \notin C_r$, then there exists an index $j \in \{1,..,n\}$ such that $|x_j| > r$. Let $\varepsilon = |x_j| - r$ and $y \in \mathbb{R}^n$ such that

$$\max_{i=1,..,n} |x_i - y_i| < \varepsilon \ ,$$

then

$$|y_j| \ = \ |y_j - x_j + x_j| \ \geq \ |x_j| - |y_j - x_j| \ > \ |x_j| - \varepsilon \ = \ r \ .$$

thus $y \notin C_r$. Hence, C_r is closed and according to Proposition 4.3, we deduce that C_r is complete.
For $n = 1$ and $\varepsilon > 0$, we have seen above that for $k > \frac{1}{\varepsilon}$ and $m > \frac{2r}{k}$

$$[-r,r] \subset \bigcup_{j=0}^{m} B(-r + \frac{j}{k}, \varepsilon) \ .$$

Therefore

$$[-r,r]^n \subset \bigcup_{j_1,...,j_n=0,...m} B_\infty((-r + \frac{j_1}{k}, .., -r + \frac{j_n}{k}), \varepsilon) \ .$$

Thus i)a) and i)b) are satisfies and the Theorem 6.6 implies the assertion (The separable dense subset is $\mathbb{Q}^n$.) ■

Theorem 6.9. *Let* $C \subset \mathbb{R}^n$ *be a subset. The following are equivalent*

1) *C is compact.*
2) *C is closed and there exists an r such that*

$$C \subset B(0,R) \ .$$

(That is C is bounded.)

Proof: 2) $\Rightarrow$ 1) Let

$$C \subset B(0,R) \subset [-R,R]^n$$

be a closed set. Since $[-R,R]^n$ is compact, we deduce from Proposition 6.4 that C is compact as well.

1) $\Rightarrow$ 2) Let C subset $\mathbb{R}^n$ be a compact set. According to Theorem 6.6 i)b), we find

$$C \subset B(x_1,1) \cup \cdots \cup B(x_m,1)$$

thus for $r = \max_{i=1,..,m}(d(x_i,0)+1)$ we have

$$C \subset B(0,r) \ .$$

Moreover, by Theorem 6.6 i)a) and Proposition 6.1, we deduce that C is closed. ■
We will now discuss one of the most important applications.

THEOREM 6.10. *Let (X,d) be a compact metric space and $f : X \to \mathbb{R}$ be a continuous function. The there exists $x_0 \in X$ such that*

$$f(x_0) \ = \ \sup\{f(x) : x \in X\} \ .$$

PROOF. Let us first assume

$$A = \{f(x) : x \in X\}$$

is bounded and $s = \sup A$. For every $n \in \mathbb{N}$, we know that $s - \frac{1}{n}$ is no upper bound. Hence there $x_n \in X$ such that

$$s \geq f(x_n) > s - \frac{1}{n} \ .$$

Let (n_k) be such that $\lim_k x_{n_k} = x \in X$. Then we deduce from continuity that

$$f(x) \ = \ \lim_k f(x_{n_k}) \ \geq \ \lim s - \frac{1}{n_k} \ = \ s \ .$$

By definition of s we find $f(x) = s$. Now, we show that A is bounded. Indeed, if note we find $x_n \in X$ such that $f(x_n) \geq n$. Again we find a convergent subsequence (x_{n_k}). Since $f(x_{n_k})$ is convergent it is bounded. We assume (f_{n_k}) is bounded above by $m \in nz$. Choosing $k \geq m+1$ we get

$$m \geq f(x_{n_k}) \geq n_k > n_m \geq m \ .$$

This contradiction shows that A is bounded and hence the first argument applies. ■

7. The theorem of Arzela-Ascoli

In the following (X, d) is a metric space.

Definition 7.1. *A subset $A \subset X$ is called relatively compact (rel.cp.), if $\bar{A}$ is compact.*

Remark 7.2. *Let (X, d) be complete and A. The following are equivalent*

i) *A is relatively compact,*
ii) *A is totally bounded,*
iii) *Every sequence (x_n) in A has a limit point (not necessarily in A.*

Proof. Follows from the characterization theorem for compact sets in metric spaces. ■

Definition 7.3. *Let (Y, d') be a metric space. A subset $F \subset C(X, Y)$ is called equi-continuous if for every $x \in X$, $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in F$, $y \in X$*

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon\ .$$

Lemma 7.4. *Let K be a compact metric space and $F \subset C(K, Y)$ be equi-continuous. Then, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in F$, $x, y \in K$*

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon\ .$$

Proof. Let $\varepsilon > 0$. For every $x \in K$, we chose δ_x such that for all $f \in F$

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \frac{\varepsilon}{2}\ .$$

By compactness there exists a finite subset $S \subset K$ such that

$$K \subset \bigcup_{x \in S} B(x, \frac{\delta_x}{2})\ .$$

We define $\delta = \min_{x \in S} \frac{\delta_x}{2}$ and assume that $d(y, z) < \delta$. Then there exists an $x \in S$ such that

$$d(x, y) < \frac{\delta_x}{2}\ .$$

Since $\delta \le \frac{\delta_x}{2}$ we find

$$d(x, z) \le d(x, y) + d(y, z) < \frac{\delta_x}{2} + \delta < \delta_x\ .$$

This implies

$$d'(f(y), f(z)) \le d'(f(y), f(x)) + d'(f(x), f(z)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

The assertion is proved. ■

LEMMA 7.5. *Let* (Y, d') *be complete metric space and* $(f_n) \subset C(K, Y)$. *Let* $D \subset K$ *be a countable set such that* $\{f_n(d) : n \in \mathbb{N}\}$ *is relative compact for every* $d \in D$. *Then there exists a subsequence* (n_k) *such that*

$$\lim_k f_{n_k}(d)$$

exists for all $d \in D$.

PROOF. We assume $D = \{d_j : j \in \mathbb{N}\}$. Since $\{f_n(d_1) : n \in \mathbb{N}\}$ is relatively compact, we may find a subsequence (n_k^1) such that

$$y_1 = \lim_k f_{n_k^1}(d_1)$$

exists and

$$d'(y_1, f_{n_k^1}(d_1)) < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Then we find a subsequence (n_k^2) of (n_k^1) such that

$$y_2 = \lim_k f_{n_k^1}(d_1)$$

exists and

$$d'(y_2, f_{n_k^2}(d_1)) < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Since (n_k^2) is a subsequence we know that

$$n_k^2 = n_{m(k)}^1$$

for some $m(k) \geq k$. Thus we also have

$$d'(y_1, f_{n_k^2}(d_1)) < \frac{1}{k}$$

for all $k \in \mathbb{N}$. We continue inductively and find successive subsequence (n_k^j) such that

$$y_j = \lim_k f_{n_k^j}(d_j)$$

exists and for all $j \leq k$ we have

$$d'(y_j, f_{n_k^j}(d_j)) \leq \frac{1}{k} .$$

Then we define the diagonal subsequence $m_k = n_k^k$. For $j \in \mathbb{N}$ we note that for $k \geq j$ we have

$$d'(y_j, f_{m_k}) = d'(y_j, f_{n_k^k}(d_j)) \leq \frac{1}{k} .$$

Hence $\lim_k f_{m_k}(d_j) = y_j$. ■

Theorem 7.6 (Arzela-Ascoli). *Let (Y, d) be a complete metric space. Let K be compact metric space. Let $F \subset C(K, Y)$ be a set. The following are equivalent*

i) *F is relatively compact.*

ii) *F is equi-continuous and or every $x \in K$ the set $\{f(x) : f \in F\}$ is relatively compact.*

Proof. *ii)* $\Rightarrow$ *i)*. Since K is a compact metric space, we know that there is a countable dense subset $D \subset K$. Let $(f_n) \subset F$ be sequence. According to Lemma 7.5 we may find a subsequence (n_k) such that

$$g(d) \;=\; \lim_k f_{n_k}(d)$$

exists for all $d \in D$. We claim that f is continuous. Indeed, we may apply Lemma 7.4. Thus, we may find for every $\varepsilon > 0$ and $\delta(\varepsilon) > 0$ such that

$$d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon \,.$$

holds for all $f \in F$. Thus given x and $y \in D$ with $d(x, y) < \delta$, we find

$$d'(g(x), g(y)) \;=\; \lim_k d'(f_{n_k}(x), f_{n_k}(y)) \le \varepsilon \,.$$

This shows that g is uniformly continuous and hence admits continuous extension $G : K \to Y$. We want to show that

$$\lim_k f_{n_k} \;=\; G \,.$$

Note that the unique extension G also satisfies

$$d(x, y) \le \delta(\varepsilon) \Rightarrow d'(G(x), G(y)) \le \varepsilon$$

(see proof of the unique extension principle). For let $\varepsilon > 0$ and $\delta = \delta(\frac{\varepsilon}{3})$ as above. Since D is dense we have $K \subset \bigcup_{d \in D} B(d, \delta)$. By compactness, we find a finite set $S \subset D$ such that

$$K \subset \bigcup_{d \in S} B(d, \delta) \,.$$

Now, we may choose k_0 such that

$$d'(G(d), f_{n_k}(d)) < \frac{\varepsilon}{3}$$

holds for all $k > k_0$. Let $x \in K$ and $y \in S$ such that

$$d(x, y) < \delta \,.$$

Then, we have

$$d'(G(x), f_{n_k}(x)) \le d'(G(x), G(d)) + d'(G(d), f_{n_k}(d)) + d'(f_{n_k}(d), f_{n_k}(x))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \; = \; \varepsilon \, .$$

This means

$$\sup_{x \in K} d'(G(x), f_{n_k}(x)) \; \leq \; \varepsilon$$

for all $k \geq k_0$.

i) $\Rightarrow$ *ii)* In the hw you will show that if $h : Z \to Y$ is a continuous map and $F \subset Z$ is relatively compact, then $h(A)$ is relatively compact. We apply this $x \in K$ and $h_x \; : \; C(K,Y) \; \to \; Y$ given by $h_x(f) \; = \; f(x)$. Thus $\{f(x) \, : \, f \in F\} = h_x(F)$ is relatively compact. Now, we show equi-continuity. Let $x \in K$ and $\varepsilon > 0$. Since F is totally bounded, there are $f_1, ..., f_m$ such that

$$F \subset \bigcup_j B(f_j, \frac{\varepsilon}{3}) \, .$$

For these function $f_1, ..., f_m$ we may find $\delta > 0$ such that $d(x,y) < \delta$ implies

$$d'(f_j(x), f_j(y)) < \frac{\varepsilon}{3}$$

for all $j = 1, ..., m$. Now, we consider $f \in F$ and $d(x,y) < \delta$. Then we may find j such that $d(f, f_j) < \frac{\varepsilon}{3}$. This implies

$$d'(f(x), f(y)) \leq d'(f(x), f_j(x)) + d(f_j(x), f_j(y)) + d'(f_j(y), f(y)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \; = \; \varepsilon \, .$$

Thus F is equi-continuous. ■