

Banach spaces-Homework 1

Due Date: January 29.

1. Let X be a Banach space. Let $J_p(X)$ be the space of all sequences such that

$$\|x\|_{J_p(X)} = \sup_{i_0 < i_1 < \dots} \left(\|x_{i_0}\|^p + \sum_{j \geq 1} \|x_{i_j} - x_{i_{j-1}}\|^p \right)^{\frac{1}{p}}$$

is finite. Show that $(J_p(X), \|\cdot\|_{J_p(X)})$ is Banach space. Also show that $J_1(X)$ and $\ell_1(X)$ are isometrically isomorphic.

2. Let $1 \leq p, q, \leq \infty$. We consider the Banach space

$$L_p \oplus_1 L_q = \{(x, y) : x \in L_p, y \in L_q\}$$

equipped with the norm

$$\|(x, y)\| = \|x\|_p + \|y\|_q.$$

Here x, y stand for equivalence classes of functions. Let $V = \{(x, x) : x \in L_p \text{ and } x \in L_q\}$. Show that there is an injective continuous embedding linear map $u : L_p \oplus L_q / V \rightarrow L_0$. Recall here that L_0 is the space of measurable functions equipped with the metric

$$d_0([f], [g]) = \inf\{\varepsilon : \mu(|f - g| > \varepsilon) < \varepsilon\}.$$

Consider the measure space $(\mathbb{R}, \Sigma, \lambda)$, where λ is the Lebesgue measure and Σ the σ -algebra of measurable functions. Show that the range of u is strictly bigger than L_p and L_q . Why does that not work for $[0, 1]$.

3. Let X, Y be Banach spaces and V be a topological vector space. Let $\iota_X : X \rightarrow V$ and $\iota_Y : Y \rightarrow V$ be injective linear maps. Show that

$$= \{v \in V : \exists_{x \in X, y \in Y} v = \iota_X(x) + \iota_Y(y)\}$$

equipped with the norm

$$\|v\|_{K_t} = \inf_{v = \iota_X(x) + \iota_Y(y)} \|x\| + t\|y\|$$

is a Banach space. What does open mapping say in this circumstances. (Hint: In the proof simply write $v = x + y$ and ignore the embedding maps). What is the dual space of K_t ?