

Homework 468

Due: 8/30

Let X be a vector space. Subset $K \subset X$ is called convex if for all $x, y \in K$ and $0 \leq \lambda \leq 1$

$$(1 - \lambda)x + \lambda y \in K$$

Let $K \subset X$ be convex, then a function $\phi : K \rightarrow \mathbb{R}$ is convex if for all $x, y \in K$ and $0 \leq \lambda \leq 1$

$$\phi((1 - \lambda)x + \lambda y) \leq (1 - \lambda)\phi(x) + \lambda\phi(y).$$

1. Let $1 \leq p < \infty$ and consider the following expression

$$\|(x_k)\|_{v_p} = \sup_{(k_j) \text{ subsequence}} \left(|x_{k_1}|^p + \sum_{j=1}^{\infty} |x_{k_{j+1}} - x_{k_j}|^p \right)^{\frac{1}{p}}.$$

Let V_p be the space of sequences such that $\|(x_k)\|_{v_p}$ is finite. Show that $(V_p, \|\cdot\|_{v_p})$ is a Banach space.

2. Let $K \subset X$ and $L \subset Y$ be convex sets and $\Phi : K \times L \rightarrow \mathbb{R}$ be convex. Show that $\phi : K \rightarrow [\infty, \infty)$ defined by

$$\phi(x) = \inf_{y \in L} \Phi(x, y)$$

is convex.

3. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be convex with $\phi(0) = 0$. Show that for $s < t$

$$\frac{\phi(s)}{s} \leq \frac{\phi(t)}{t}.$$

Show that

$$\phi'_+(0) = \lim_{t>0, t \rightarrow 0} \frac{\phi(t)}{t}$$

exists and

$$\phi(t) \geq t\phi'_+(0).$$

4. Let X be a vector space $f : X \rightarrow \mathbb{R}$ be convex, $V \subset X$ be a subspace and $l : V \rightarrow \mathbb{R}$ be a linear map such that $l \leq f$ on V . Let $x \in X \setminus V$ and V_1 be the subspace of X spanned by V and x . Show that there exists an extension l_1 of l to V_1 such that still $l_1 \leq f$ on V_1 . (Hint consider the map $\Phi : \mathbb{R} \times V \rightarrow \mathbb{R}$ defined by $\Phi(t, y) = f(tx + y) - l(y)$.)

Math 468-Banach spaces, No 2.

Due: 9/6/2000

1. Show that

$$d(\ell_1^n, \ell_2^n) \leq \sqrt{n}.$$

2. Show that for two n -dimensional spaces E and F

$$d(E, F) = d(E^*, F^*).$$

3. Let $1 \leq p, q \leq \infty$ and α be the norm on matrices defined by

$$\alpha(a) = \left(\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{p}}.$$

Calculate α^* .

4. Let α be the norm of $n \times n$ matrices given by

$$\alpha(a) = \sum_{j=1}^k b_j$$

where $1 \leq k \leq n^2$ and b_1, \dots, b_{n^2} provides the list of elements $\{|a_{11}|, \dots, |a_{nn}|\}$ such that $b_1 \geq b_2 \geq \dots \geq b_{n^2}$. Calculate α^* .

5. Prove the arithmetic/ geometric mean inequality

$$\left(\prod_{j=1}^n a_j \right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{j=1}^n a_j$$

and discuss equality. (Hint: $f(x) = -\log(x)$ is convex.)

Math 468-Banach spaces, No 3.

Due: 9/13/2000

Notation: Let $(x_j) \subset X$ be a sequence in a Banach space X . We put

$$\|(x_j)\|_p = \left(\sum_{j=1}^{\infty} \|x_j\|_X^p \right)^{\frac{1}{p}} .$$

and

$$w_p((x_j)) = \sup_{\|x^*\|_{X^*} \leq 1} \left(\sum_{j=1}^{\infty} |x^*(x_j)|^p \right)^{\frac{1}{p}} .$$

Let $\frac{1}{p} + \frac{1}{p'} + 1$. For a linear map $T : X \rightarrow Y$, we put

$$\nu_p(T) = \inf \|(x_j^*)\|_{p'}, w_p((y_j)) ,$$

where the infimum is taken over all sequences $(x_j^*) \subset X^*$, $(y_j) \subset X$ such that

$$T(x) = \sum_{j=1}^{\infty} x_j^*(x) y_j . \tag{1}$$

We denote by $N_p(X, Y)$ the set of linear maps $T : X \rightarrow Y$ such that $\nu_p(T) < \infty$.

1. Show that $(N_p(X, Y), \nu_p)$ is a Banach space. (Hint: Use $st = \inf_{r>0} \frac{1}{p}(rs)^p + \frac{1}{p'}(rt^{-1})^{p'}$ and the fact that for $T \in N_p(X, Y)$, $\varepsilon > 0$, one can choose (x_j^*) , (y_j) satisfying (1) such that

$$\|(x_j^*)\|_{p'}^{p'} = w_p((x_j))^p \leq (1 + \varepsilon) \nu_p(T) .$$

2. (a) Let $x_1, \dots, x_n \in \ell_2$. Show that

$$\frac{1}{2^n} \sum_{\varepsilon_1 = \pm 1, \dots, \varepsilon_n = \pm 1} \left\| \sum_{k=1}^n \varepsilon_k x_k \right\|_2^2 = \sum_{k=1}^n \|x_k\|_2^2 .$$

- (b) Let X be a Banach space and $T : \ell_{\infty}^n \rightarrow X$ be a linear map. Show that

$$\sum_{j=1}^n \|T(x_j)\|_X = \sup_{\|S : X \rightarrow \ell_{\infty}^n\| \leq 1} |tr(TS)| .$$

(c) Let $T : \ell_\infty^n \rightarrow \ell_2^n$. Show that

$$\sum_{j=1}^n \|T(e_j)\|_2 \leq \sqrt{n} \|T\| .$$

(d) Show

$$d(\ell_2^n, \ell_\infty^n) \geq \sqrt{n}.$$

Math 468-Banach spaces, No 4.

Due: 9/20/2000

1. Let $T : X \rightarrow X$ be a finite rank map and let $(x_j^*)_{j=1}^m, (x_j)_{j=1}^m, (y_j^*)_{j=1}^n, (y_j)_{j=1}^n$, such that for all $x \in X$

$$T(x) = \sum_{j=1}^m x_j^*(x)x_j \quad \text{and} \quad T(x) = \sum_{j=1}^n y_j^*(x)y_j .$$

Show

$$\sum_{j=1}^m x_j^*(x_j) = \sum_{j=1}^n y_j^*(y_j) .$$

2. Let $\alpha = \sum_j \alpha_j$ be summable sequence of real numbers and $\varepsilon > 0$. Find a sequence (η_j) such that

$$\lim_j \eta_j = 0 \quad \text{and} \quad \sum_j \eta_j^{-1} \alpha_j \leq (1 + \varepsilon)\alpha .$$

3. On \mathbb{K}^n we define the norm

$$\|x\| = (a_1^2 + a_2^2)^{\frac{1}{2}} ,$$

a_1, \dots, a_n is the non-decreasing arrangement of $\{|x_1|, \dots, |x_n|\}$, i.e. $a_1 \geq a_2 \geq \dots \geq a_n$ and all the values $|x_1|, \dots, |x_n|$ appear in that list. Show that $\|\cdot\|$ is a norm. Find the intersection of this unit ball with the $x = y$ plane. Calculate the norm of the functional $\phi : (\mathbb{R}^3, \|\cdot\|) \rightarrow \mathbb{R}, \phi(x) = x_1 + x_2 + x_3$.

1. Let $(x_n) \subset X$ be a sequence in a Banach space such that

$$\lim_n \|x_n\| = 0.$$

Show that the absolute convex hull

$$\left\{ \sum_n \alpha_n x_n : \sum_n |\alpha_n| \leq 1 \right\}$$

is compact.

2. Let $C \subset X$ be a compact set

- (a) Show that for every $\varepsilon > 0$ there exists $x_1, \dots, x_N \in C$ such that for every $x \in C$ there exists an $i \in \{1, \dots, N\}$ such that

$$\|2x - x_i\| \leq \varepsilon.$$

- (b) Let C and x_1, \dots, x_N be chosen as above for $\varepsilon = \frac{1}{4}$. Show that

$$C_2 = \bigcup_{i=1}^N ((B(x_i, \frac{1}{4}) \cap 2C) - x_i)$$

is again compact.

- (c) Show that every compact set is in the closed convex hull of a sequence converging to 0.

Math 468-Banach spaces, No 6.

Due: 9/4/2000

1. Show that

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \sum_{i=1}^n x_i^* y_i^*$$

Where $(x_1^*, \dots, x_n^*), (y_1^*, \dots, y_n^*)$, is the non-decreasing rearrangement of $(|x_1|, \dots, |x_n|), (|y_1|, \dots, |y_n|)$, respectively. Hints: There is no loss of generality to assume that $x_1 = x_1^*, \dots$, the same for the y_1, \dots, y_n and the for all permutations we have to show

$$\sum_{i=1}^n x_i y_{\pi(i)} \leq \sum_{i=1}^n x_i y_i$$

Observe that for any π

$$\sum_{j=1}^k y_{\pi(j)} \leq \sum_{j=1}^k y_j$$

and use the Abel summation

$$\sum_{i=1}^n x_i y_{\pi(i)} = \sum_{i=1}^n x_i \left(\sum_{j=1}^i y_{\pi(j)} - \sum_{j=1}^{i-1} y_{\pi(j)} \right) = \dots$$

2. Let α be the norm of $n \times n$ matrices given by

$$\alpha_k(a) = \sum_{j=1}^k b_j$$

where $1 \leq k \leq n^2$ and b_1, \dots, b_{n^2} provides the list of elements $\{|a_{11}|, \dots, |a_{nn}|\}$ such that $b_1 \geq b_2 \geq \dots \geq b_{n^2}$. Calculate α_k^* .

Math 468-Banach spaces, No 7.

Due: 10/11/2000

1. Let (Ω, Σ, μ) be a measure space and A_1, \dots, A_m be disjoint sets in Σ such that $\mu(A_i) > 0$ for $i = 1, \dots, m$. We define the map

$$E(f)(\omega) = \sum_{i=1}^m \int_{A_i} f \frac{d\mu}{\mu(A_i)} 1_{A_i}(\omega).$$

Recall

$$1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases}.$$

Show for $1 \leq p \leq \infty$ and $f \in L_p(\Omega, \Sigma, \mu)$

$$\|E(f)\|_p \leq \|f\|_p.$$

Let

$$X_p = \left\{ \sum_{i=1}^m \alpha_i 1_{A_i} : \alpha_i \in \mathbb{K} \right\} \subset L_p(\Omega, \Sigma, \mu).$$

Show that

$$\|f - E(f)\|_p = \inf\{\|f - g\|_p : g \in X_p\}.$$

2. Let X, Y Banach spaces. We say that a net (T_i) converges to T in the topology τ if for every compact set $K \subset X$

$$\limsup_i \sup_{x \in K} \|T_i(x) - T(x)\|_Y = 0.$$

Let $(y_n)^* \subset Y^*$ and $(x_n) \subset X$ such that

$$\sum_n \|y_n^*\|_{Y^*} \|x_n\|_X < \infty.$$

Show that

$$\phi(T) = \sum_n y_n^*(T(x_n))$$

is continuous with respect to τ . (Hint you can assume that $\lim_n \|x_n\| = 0$, why?). Let $K \subset X$ be a compact set and (x_n) a sequence tending to 0 such that $K = \overline{\text{conv}}\{x_n\}$ (Why can we assume that?). Consider

$$\Psi : L(X, Y) \rightarrow c_0(Y)$$

defined by

$$\Psi(T) = (T(x_n)) .$$

Show that

$$\|\Psi(T)\| = \sup_{x \in K} \|Tx\| .$$

Use the Hahn-Banach theorem to show that for every functional $\phi : \mathcal{L}(X, Y) \rightarrow \mathbb{K}$ with

$$|\phi(T)| \leq \sup_{x \in K} \|Tx\|$$

there are $(y_n^*) \subset Y^*$ such that for all T

$$\phi(T) = \sum_n y_n^*(T(x_n))$$

and

$$\sum_n \|y_n^*\| < \infty .$$

(Hint, $c_0(Y)^* = \ell_1(Y^*)$.)

3. Let X, Y be Banach spaces. Let $X^* \subset C(B_{X^{**}}), Y \subset C(B_{Y^*})$ be the canonical embeddings. For a finite rank map $S = \sum_{i=1}^n x_i^* \otimes y_i$, we define

$$\Psi(S)(x^{**}, y^*) = \sum_{i=1}^n x^{**}(x_i) y^*(y_i) .$$

Show that

$$\|\Psi(S)\|_{C(B_{X^{**}} \times B_{Y^*})} = \|S\| .$$

What does the Hahn-Banach tell you about an integral map $T : Y \rightarrow X$?

Math 468-Banach spaces, No 8.

Due: 10/18/2000

1. Let $1 \leq p < q \leq \infty$ and $T : X \rightarrow Y$ be an absolutely p -summing map. Show that

$$\pi_q(T) \leq \pi_p(T).$$

(Hint: Choose r such that $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and observe that for $x_1, \dots, x_n \in X$

$$\left(\sum_{k=1}^n \|T(x_k)\|_Y^q \right)^{\frac{1}{q}} = \sup_{\sum_k |\tau_k|^r \leq 1} \left(\sum_{k=1}^n \|T(\tau_k x_k)\|_Y^p \right)^{\frac{1}{p}},$$

where the supremum is taken over positive real numbers τ_1, \dots, τ_n .

2. Let $2 \leq p \leq \infty$ and $i_{2,p}^n : \ell_2^n \rightarrow \ell_p^n$ be the identity map. Show

$$\pi_2(i_{2,p}^n) = \sqrt{n} \quad \text{and} \quad \pi_2((i_{2,p}^n)^{-1}) = \sqrt{n}.$$

Find the ellipsoid of maximal volume in the unit ball of ℓ_p^n .

3. Let $\mathbb{K} = \mathbb{R}$ and $i_{1,2}^n : \ell_1^n \rightarrow \ell_2^n$ be the identity map. Show

$$\pi_2(i_{1,2}^n) = 1.$$

(Hint: Consider $S = \{-1, 1\}^n$ and $\mu(\{\varepsilon_1, \dots, \varepsilon_n\}) = 2^{-n}$ the normalized counting measure. Then the map $i(x)(\varepsilon) = \sum_{i=1}^n \varepsilon_i x_i$ is an isometric embedding of ℓ_1^n in $L_\infty(S, \mu)$.)

Math 468-Banach spaces, No 9.

Due: 10/25/2000

1. Our goal is to show in the real case

$$\pi_1(id_{\ell_1^n}) > \sqrt{n}.$$

- (a) Let ν be the Haar measure on $\{-1, 1\}^n$, i.e.

$$\nu(\{(\varepsilon_1, \dots, \varepsilon_n)\}) = 2^{-n}.$$

Show that

$$\int_{\{-1,1\}^n} \left| \sum_{i=1}^n \varepsilon_i \right| d\nu(\varepsilon) < \sqrt{n}.$$

- (b) Let $c = \pi_1(id_{\ell_1^n})$. Show that c is the smallest constant such that for all $x_1, \dots, x_n \in \mathbb{R}$

$$\sum_{i=1}^n |x_i| \leq c \int_{\{-1,1\}^n} \left| \sum_{i=1}^n x_i \varepsilon_i \right| d\nu(\varepsilon).$$

(Hint: By averaging an arbitrary measure can be replaced by the Haar measure and ℓ_1^n is isometrically embedded in $\ell_\infty(\{-1, 1\}^n)$).

- (c) Conclude $\pi_1(id_{\ell_1^n}) > \sqrt{n}$.

2. Show that in the real case $\pi_1(id_{\ell_2^n})$ is the smallest constant c such that for all $x_1, \dots, x_n \in \mathbb{R}$

$$\left(\sum_{i=1}^n |x_i|^2 \right)^{\frac{1}{2}} \leq c \int_{S^{n-1}} \left| \sum_{i=1}^n x_i y_i \right| d\sigma(y),$$

where σ is the rotation invariant normalized surface measure on the sphere S^{n-1} . Conclude again $\pi_1(id_{\ell_2^n}) > \sqrt{n}$.

3. Let (α_n) be a sequence in ℓ_2 of norm less than one. Show that there exists a Banach space X and vectors $(x_n) \subset X$ such that

$$\alpha_n = \|x_n\|_X$$

and

$$\sup_{\varepsilon_n = \pm 1} \left\| \sum_n \varepsilon_n x_n \right\|_X \leq 1.$$

Math 468-Banach spaces, No 10.

Due: 11/1/2000

1. Let $2 \leq p < \infty$. A subset $S \subset \mathbb{Z}$ is called Λ_p set if there exists a constant $c > 0$ such that for all finite sequences of scalars

$$\left(\int_0^{2\pi} \left| \sum_{k \in \Lambda} \alpha_k e^{ikt} \right|^p \frac{dt}{2\pi} \right)^{\frac{1}{p}} \leq c \left(\sum_{k \in \Lambda} |\alpha_k|^2 \right)^{\frac{1}{2}} .$$

Then $\Lambda_p(S) = \inf c$, the infimum is taken over all possible constant. Let $P_\Lambda : C([0, 2\pi]) \rightarrow \ell_2(S)$ be the map

$$P_S(f) = (\hat{f}(k))_{k \in S} ,$$

where

$$\hat{f}(k) = \int_0^{2\pi} f(t) e^{-ikt} \frac{dt}{2\pi} .$$

Let $1 \leq p' \leq 2$ be defined by $\frac{1}{p} + \frac{1}{p'} = 1$. Show that

$$\Lambda_p(S) = \Pi_{p'}(P_S) .$$

2. Show that for all finite sequences (α_k)

$$\left(2^{-n} \sum_{\varepsilon_k = \pm 1} \left| \sum_{k=1}^n \alpha_k \varepsilon_k \right|^4 \right)^{\frac{1}{4}} \leq 3 \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}} .$$

(I hope 3 is okay it might be 5, though)

Math 468-Banach spaces, No 10.

Due: 11/8/2000

1. Let h_1, \dots, h_n be positive, measurable functions such that for all $i = 1, \dots, n$

$$\int_{\mathbb{R}} h_i(x) dx = 1$$

and

$$\int_{\mathbb{R}} |x| h_i(x) dx < \infty$$

and $h_i(x) = h_i(-x)$. Let y_1, \dots, y_n be vectors in a Banach space Y . Show

$$\begin{aligned} & \int_{\mathbb{R}^n} \sup_{i=1, \dots, n} |x_i| \|y_i\| h_1(x_1) \cdots h_n(x_n) dx_1, \dots, dx_n \\ & \leq \int_{\mathbb{R}^n} \left\| \sum_{i=1}^n x_i y_i \right\|_Y h_1(x_1) \cdots h_n(x_n) dx_1, \dots, dx_n. \end{aligned}$$

2. Let h_1, \dots, h_n be as above and

$$\gamma_p(i) = \left(\int_{\mathbb{R}} |x|^p h_i(x) dx \right)^{\frac{1}{p}} < \infty$$

and

$$\gamma_p = \sup_{i=1, \dots, n} \gamma_p(i).$$

Show

$$\int_{\mathbb{R}^n} \sup_{i=1, \dots, n} |x_i| h_1(x_1) \cdots h_n(x_n) dx_1, \dots, dx_n \leq n^{\frac{1}{p}} \gamma_p.$$

Math 468-Banach spaces, No 12.

Due: 11/15/2000

1. $n, m \in \mathbb{N}$ Show that

$$\int \left\| \sum_{i=1, \dots, n, j=1, \dots, m} x_{ij} e_i \otimes e_j : \ell_2^n \rightarrow \ell_2^m \right\| d\gamma_{nm}(x) \leq C\sqrt{n+m}.$$

Hint: Using Δ -nets this follows from the deviation inequality considering the random variables

$$F_{y,z}(x) = \left| \sum_{ij} x_{ij} z_i y_j \right|.$$

2. Let $2 \leq p < \infty$ Show

$$\int \sup_{k=1, \dots, n} x_k^* d\gamma(x) \sim_{c(p)} n^{\frac{1}{p}}.$$