1 Finite dimensional spaces

Recall: all finite dimensional Banach space with fixed dimension "coincide"

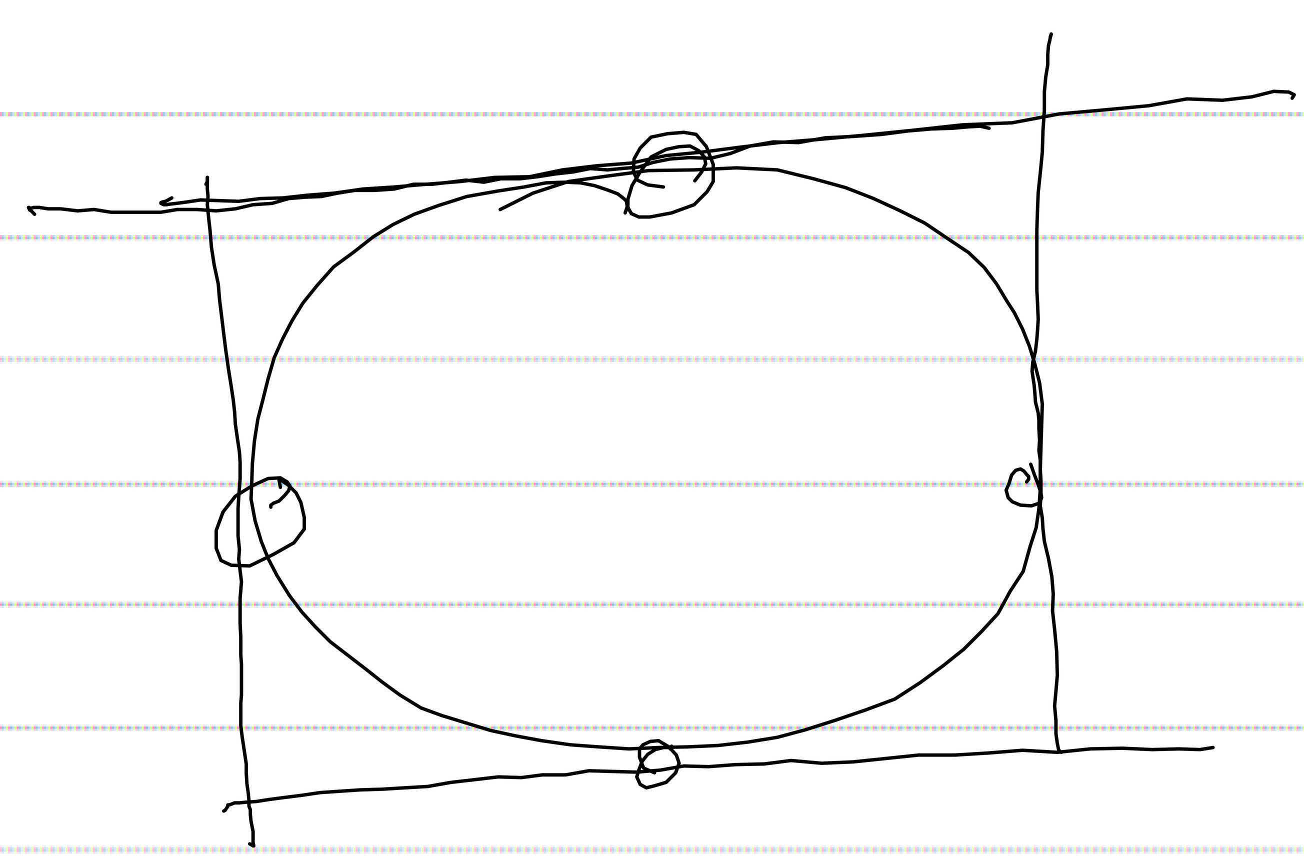
or all norms on $\mathbb{R}^n$ are equivalent.

We want to prove something more precise

Theorem Let E be a n-dimensional Banach space. Then there exists vectors $x_1 \dots x_n \in E$, $x_1^* \dots x_n^* \in E$ such that

$$x_j^*(x_k) = \delta_{jk} \quad \|x_j^*\| = 1 = \|x_k\|$$

Geometrically



$((x_j), (x_j^*))$ is called Auerbach basis.

Determinants

a is a $n \times n$ matrix

$$\det a = \sum_{\sigma \in \mathrm{Perm}_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} a_{j\sigma(j)} \qquad (*)$$

has the following properties

$\det ab = \det a \det b$

$\det a = 0 \iff a$ has not full rank

$\det a = \prod_{j=1}^{n} \lambda_j(a)$ if a has n eigenvalues

$\det 1 = 1$

$$\boxed{\begin{array}{l} f(t) = \det(1 + ta) \text{ satisfies} \\ f'(0) = \mathrm{tr}(a) = \sum_{k=1}^{n} a_{kk} \end{array}}$$ follows easily

(other properties not so easy to obtain from $*$)

Def let α be a norm on $M_n = n \times n$ matrices

$$\alpha^*(a) = \sup \{ |\mathrm{tr}(ab)| \mid \alpha(b) \le 1 \}$$

$\mathrm{tr}(ab) = \sum_{kl} a_{lk} b_{kl}$ (up to $(l,k) \to (k,l)$ standard dual)

Lemma (Lewis) Let α be norm on M_n. Then there exist $a_0 \in M_n$ invertible such that

$$\alpha(a_0) = 1 \qquad \alpha^\vee(a_0^{-1}) = n$$

Proof: a_0 such that

$$|\det(a_0)| = \sup\{ |\det(a)| : \alpha(a) \leq 1 \}$$

Then $|\det(a_0)| \geq \dfrac{|\det(1)|}{\alpha(Id)^n} > 0$ hence invertible.

$$|\det(a_0 + tb)| \leq \alpha(a_0 + tb)^n \, |\det(a_0)|$$

$$\Rightarrow |\det(1 + t a_0^{-1} b)| \leq (1 + |t|\alpha(b))^n$$

$$\| \qquad\qquad = 1 + |t| n \alpha(b) + O(t^2)$$

$$|1 + t \operatorname{tr}(a_0^{-1} b) + t^2 \,| \qquad \boxed{t \in K}$$

We may choose $t = |t| \operatorname{sgn}(t)$ such that $\operatorname{sgn}(t) \operatorname{tr}(a_0^{-1} b) \geq 0$

Hence $1 + t|\operatorname{tr}(a_0^{-1}b)| + O(t^2) \leq 1 + t n \alpha(\operatorname{sgn}(t) b) + o(t^2)$

Then $|\operatorname{tr}(a_0^{-1}b)| \leq n \alpha(b) + o(|t|) \xleftarrow{|t| \to 0}$

$|\operatorname{tr}(a_0^{-1}b)| \leq n \alpha(b)$ Then $\boxed{\alpha^\vee(a_0^{-1}) \leq n.}$ ∎

Application 1: There exists an Auerbach basis

Proof $\alpha(a) = \sup_{i=1,\dots,n} \|a(e_i)\|_E$ $\quad a\ \mathbb{R}^n \to E = (\mathbb{R}^n, \|\ \|)$

$a(e_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$ x_i

We find a invertible such that

$x_i = a(e_i)$ satisfy $\|x_i\| = 1$

and $\alpha^*(a^{-1}) = n$

What is dual norm

$$\sup_{\alpha(a)\le 1} \Big|\sum_{ik} b_{ik} a_{ki}\Big| = \sup_{\|x_i\|\le 1} \Big|\sum_{i=1}^{n} \langle y_i, x_i\rangle\Big|$$

$$= \sum_{i=1}^{n} \|y_i\|_E^*$$

where $y_i \in E^*$ is given by

$\quad\ \ \shortparallel$

$(b_{i1}, \dots, b_{in})$

Hence we find $y_i \in E^*$

$$n = \sum_{i=1}^{n} \|y_i\|_E^* = \sum_{i=1}^{n} \langle y_i, x_i\rangle = \mathrm{tr}(\mathrm{Id})$$

Hence $\langle y_i, x_i\rangle = \|y_i\|_E^*$ for all $i=1,\dots,n$
and $\|x_i\| = 1$ for all $i=1,\dots,n$.

Moreover $\langle y_j, x_i\rangle = \delta_{ij}$ because we are working with inverse. Therefore

$$\|y_i\|_E^* \geq \frac{\langle y_i, x_i\rangle}{\|x_i\|} \geq 1$$

If $\|y_j\|_{E^*} > 1$ then $\sum \|y_i\|_E^* > n$ $\not\ell$.

Thus $\|y_i\|_{E^*} = 1$ for all $i = 1 \dots n$ $\square$

Rem complex case works analogously for Lewis lemma and Auerbach basis.

Defenition A norm α on M_n has enough symmetries if there exists a compact group of isometries such that

i) $\alpha(g^{-1}ag) = \alpha(a)$

ii) $\int_G g^{-1}ag \, d\mu(g) = \frac{tr(a)}{n} \mathrm{Id}$ ← Identity.

Lemma $\alpha(\mathrm{Id})\alpha^*(\mathrm{Id}) = n$

Proof let $b \in M_n$ such that $\boxed{tr(b) = \alpha^*(\mathrm{Id})}$

$\alpha(b) = 1$

(exists by Hahn Banach) Then

$\alpha(g^{-1}bg) \leq 1$, $\int \alpha(g^{-1}bg) \leq 1$

$\qquad\qquad\qquad\qquad \text{VI}$

$\qquad\qquad\qquad \alpha(\int g^{-1}bg) \leq 1$

By property ii) we know that

$$\int g^{-1} b g = \tau_n(b)\, Id$$

and $\operatorname{tr}\left(\int g^{-1} b g\right) \cancel{\alpha^*(Id)} = \tau_n(b) \operatorname{tr}(Id) =$

$$\int \operatorname{tr}(g^{-1} b g) = \operatorname{tr}(b) = \alpha^*(Id)$$

Therefore $\lambda = \tau_n(b)$ satisfies

$$\alpha(\lambda\, Id) \le 1$$

$$|\lambda|\, \alpha(Id)$$

$$\frac{\operatorname{tr}(b)}{n}\alpha(Id) = \frac{\alpha^*(Id)\,\alpha(Id)}{n} \geq \frac{\operatorname{tr}(Id)}{n} \geq 1$$

we have equality, and the claim follows. $\blacksquare$

Two reasons for existence of $\alpha(a)\,\alpha^*(a^{-1}) = n$

Question: Uniqueness?

Lemma α a norm such that

$\alpha(au) = \alpha(a)$ for all orthogonal unitary matrices.

Then $\alpha(a)\,\alpha^*(a^{-1}) = n$ and $\alpha(a) = 1$

$\alpha(b)\,\alpha^*(b^{-1}) = n$ and $\alpha(b) = 1$

Implies that $b = au$ for some u.

Proof: Wlog we may assume that a satisfies

$\sup_{\alpha(\tilde{a}) \le 1} |\det \tilde{a}| = |\det a|$.

Consider $c = b^{-1}a$ (invertible)

Using Gram-Schmidt we may assume

$c = \nabla u$ $\quad$ u orthogonal

∇ $\quad$ upper diagonal

Note that $|\det c| = |\det b^{-1} a| = |\det b|^{-1} |\det a| \geq 1$

$$\prod_{k=1}^{n} |\nabla_{kk}| \underset{\substack{\uparrow \\ \text{geom/arithm mean}}}{\le} \left(\frac{\sum_k |\nabla_{kk}|}{n}\right)^n = \left(\frac{\operatorname{tr}(D_\varepsilon \nabla)}{n}\right)^n$$

some diagonal with $|\varepsilon_i| = 1$

$$= \left(\frac{\operatorname{tr}(D_\varepsilon c u^{-1})}{n}\right)^n = \left(\frac{\operatorname{tr}(c u^{-1} D_\varepsilon)}{n}\right)^n$$

Howewer

$$|\mathrm{tr}(b^{-1} a u^{-1} D_\varepsilon)| \le \alpha^*(b^{-1})\ \alpha(a u^{-1} D_\varepsilon)$$

$$\le n \qquad \alpha(a) \qquad \le n$$

Hence we have

$$n \le |\mathrm{tr}(b^{-1} a u^{-1} D_\varepsilon)| \le \alpha^*(b^{-1})\,\alpha(a) \le n$$

$\mathbb{K} = \mathbb{C}$ Instead of Gram-Schmidt

we use polar decomposition

$$b^{-1}a = w\ \Pi\big(\lambda_j(|b^{-1}a|)\big)\ u$$

and find

$$|\det(b^{-1}a)| = \Big(\prod \lambda_j(|b^{-1}a|)\Big)^{1/n} \geq 1$$

$$\wedge|$$

$$\frac{\sum \lambda_j(|b^{-1}a|)}{n} = \frac{|\mathrm{tr}(b^{-1}auw)|}{n} \le 1$$

Equality here implies $\lambda_j(|b^{-1}a|) = 1$ hence

$b^{-1}a$ is already a unitary.

$\mathbb{K} = \mathbb{R}$ $\quad b^{-1}a = \nabla u$ ~~. ∇ is also complex matrix~~

~~and we still have~~ $\cancel{1 \le |\det \nabla|^{1/n} \le \frac{\mathrm{tr}(|\nabla|)}{n} \le 1}$

We shall now that $\sqrt{(b^{-1}a)^*(b^{-1}a)}$ is real and

$|b^{-1}a|$ is real and $\| \, |b^{-1}a| e \| = \| b^{-1}a e\|$

Hence $b^{-1}a = |b^{-1}a| o$ orthogonal

Hence argument from above shows that

$|b^{-1}a|$ is unitary and hence orthogonal and hence $b^{-1}a$ is orthogonal

$\boxed{\equiv}$

Interpretation: The ellipsoid

$\mathcal{E} = a(B_2^n) \qquad B_2^n = \{x \in \mathbb{R}^n \, , \, \|x\|_2 \le 1\}$

is independent of the choice of a with

$\alpha(a) = 1 \qquad \alpha^*(a^{-1}) = n$

Application: The ellipsoid $\mathcal{E} \subset B_E = \{x : \|x\|_E \le 1\}$

of maximal volume satisfies

$\alpha^*(u^{-1}) = n$ where $\alpha = \|\cdot\|$

Question What is α^* ?

Same remark holds for ellipsoid of minimal volume containing B_E. Why?

Some examples

$\ell_p^n = (\mathbb{K}^n, \|\cdot\|_p)$

$\|x\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$

$\alpha(u) = \|u : \ell_2^n \to \ell_p^m\|$

Note α is a right ideal norm

$\alpha(uw) \leq \alpha(u) \, \|w \, \ell_2^n \to \ell_2^n\|$.

$G = \{-1,1\}^n \times \mathrm{Perm}_n \subseteq \mathcal{U}(\ell_2^n) =$ unitary group

$\varepsilon \longmapsto D_\varepsilon = \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{pmatrix}$

$\pi \longmapsto M_\pi (e_i) = e_{\pi(i)}$ permutation matrix

Note $D_\varepsilon M_\pi D_\varepsilon (e_i) = \varepsilon_{\pi(i)} \varepsilon_i e_{\pi(i)}$

$= D_{\tilde{\varepsilon}_\pi} M_\pi$

$\varepsilon_\pi(j) = \varepsilon_j \varepsilon_{\pi^{-1}(j)}$

Hence $\{D_\varepsilon M_\pi : \varepsilon \in \{-1,1\}, \pi \in M_\pi\}$ is indeed a subgroup. And Haar measure = counting meas

We need

$$\alpha(g^{-1}ug) = \alpha(u)$$

Indeed Π_π and D_ε are isometries on ℓ_p^n and hence $\alpha(g^{-1}ug) = \alpha(u)$ (using left multiplication.)

<u>Thm</u> The ellipsoid of maximal volume in ℓ_p^n is given by

$$B_2^n = B_{\ell_2^n} \subseteq B_{\ell_p^n} \quad \text{for } p \geq 2$$

$$n^{1/2 - 1/p} B_2^n \subseteq B_{\ell_p^n} \quad \text{for } 1 \leq p \leq 2$$

<u>Proof</u> We know that $\alpha(Id)\alpha^*(Id) = n$

Thus it suffices to calculate

$$\alpha(Id : \ell_2^n \to \ell_p^n) = \| Id\, \ell_2^n \to \ell_p^n \|$$

For $p \geq 2$ we have $\left(\sum |x_i|^p\right)^{1/p} \leq \left(\sum |x_i|^2\right)^{1/2}$

and this is attained for $x = (1, 0 \text{—} 0)$

For $1 \leq p \leq 2$ we have $\left(\sum |x_i|^p\right)^{1/p} \leq n^{1/p - 1/2} \left(\sum |x_i|^2\right)^{1/2}$

and this is attained for $x = (1 \text{—} 1)$ $\square$

Question? what is optimal u? (in following sense)

$$d(\ell_2^n,\ell_p^n) = dist(\ell_2^n,\ell_p^n) = \inf\{\|u\|\,\|u^{-1}\| : u \text{ isomorphism}\}$$

The examples above suggest to simply take

$u = Id$ and then we get

$$\boxed{d(\ell_2^n,\ell_p^n) \le n^{|1/2-1/p|}}$$

How can we show this is sharp $\longrightarrow$ Type, Cotype

<u>Definition</u> An <u>ideal</u> of norms is an assignement

$$\alpha : (E,F) \longrightarrow \alpha(E,F) \subseteq \mathcal{L}(E,F)$$

Such that

- $\alpha(E,F)$ is a set for fixed E,F
- $\alpha(E,F)$ carries a norm α
- $\alpha(RST) \le \|R\|\,\alpha(S)\,\|T\|$

  and $R \in \mathcal{L}(F,G)$ $S \in \alpha(E,F)$ $T \in \mathcal{L}(D,E)$

  implies $RST \in \alpha(E,F)$
- For any rank one map

  $T_{e^*f}(e) = e^*(e) f$ one has

  $\alpha(T_{e^*f}) = \|e^*\|\,\|f\|,$

Proposition Let α be an ideal

Let $1 \le p, q \le n$ Then the ellipsoid of maximal volume for $\alpha(\ell_p^n, \ell_q^n)$ is given by

$$\frac{B_{\ell_2^{n^2}}}{\alpha(\mathrm{id}: \ell_2^{n^2} \to \alpha(\ell_p^n, \ell_q^n))}$$

Proof: We have to identify a suitable group

$G = G_1 \times G_1 \subseteq \alpha(\ell_p^n, \ell_q^n)$

where $G_1 = \{D_\varepsilon M_\pi : \varepsilon \in \{-1,1\}^n, \pi \in \mathrm{Perm}\}$

$\pi(D_\varepsilon M_\pi, D_{\tilde\varepsilon} M_{\tilde\pi})(R) = D_{\tilde\varepsilon} M_{\tilde\pi} R D_\varepsilon M_\pi$

Obviously

$\beta(T: \ell_2^{n^2} \to \alpha(\ell_p^n, \ell_q^n))$ satisfies

$\beta(g^{-1} T g) = \beta(T)$

It suffices to prove $\int_G g^{-1} T g \, dg = \tau(T) \, \mathrm{Id}$.

has dim n^2

Let $T\ \ell_2^{n^2} \longrightarrow \mathcal{L}(\ell_p^n, \ell_q^n)$

$$T(e_{ij}) = \sum t_{ijkl}\, e_{kl}$$

assume

$\int g^{-1} T g \, dg = T$

Then $T(\varepsilon_i \tilde{\varepsilon}_j e_{ij}) = \varepsilon_i \tilde{\varepsilon}_j \sum_{kl} t_{ijkl}\, e_{kl}$

$\quad \parallel$

$g\, T(e_{ij}) = \cancel{\varepsilon_k} \sum_{ij} t_{ijkl}\, \varepsilon_k \tilde{\varepsilon}_l\, e_{kl}$

$i \neq k \qquad \varepsilon_i = \begin{cases} -1 & i \\ 1 & \text{all other} \end{cases} \qquad \tilde{\varepsilon}_j = 1$

Then we get $\quad - \sum_{kl} t_{ijkl}\, e_{kl} = \sum_{\substack{kl \\ k \neq l}} t_{ijkl}\, e_{kl} - t_{ijil}\, e_{il}$

$$\sum_{k \neq i\, l} 2 t_{ijkl}\, e_{kl} = 0$$

$\Rightarrow \quad t_{ijkl} = 0 \quad \text{if } i \neq k$

Similarly

$$t_{ijkl} = 0 \quad \text{if } j \neq l$$

$T(e_{ij}) = t_{ij}\, e_{ij}$

$T(M_\pi(e_{ij})) = T(e_{\pi(i),j}) = t_{\pi(i),j}\, e_{\pi(i),j}$

$\parallel$

$M_\pi T(e_{ij}) = t_{ij}\, e_{\pi(i),j}$

Hence $t_{\pi(i),j} = t_{ij}$ for all i

Similarly $t_{i,\pi(j)} = t_{ij}$

This means $t_{ij} = t_{11}$ is a constant

$$\boxed{T = t_{11}\ \mathrm{Id}}$$ $\square$

<u>Similar situation</u>

For matrices define

$$\|a\|_p = \left(\sum s_j(|a|)^p\right)^{1/p}$$

This space is called S_p^n

where $s_j(a) = d_j(|a|)$ $\quad |a| = \sqrt{a^* a}$

are the singular numbers

<u>Prop</u> The ellipsoid of maximal volume in S_p^n is

$$B_{\ell_2^{n^2}} \subseteq B_{S_p^n} \qquad p \geq 2$$

$$n^{1/2 - 1/p} B_{\ell_2^{n^2}} \subseteq B_{S_p^n} \qquad 1 \leq p \leq 2$$

Proof HW. $\square$