$$= \left[\frac{1}{q} \sum_{j=1}^{n} \|y_j^*\|^q + \frac{1}{q'} \sup_{x^*} \sum_{j=1}^{n} |x^*(x_j)|^{q'} \right] + \left[\quad \right]$$

$$= \left(\sum_{1}^{n} \|y_j^*\|^q\right)^{1/q} w_{q'}(x_1 \dots x_n) + \left(\sum \|y_j^*\|^q\right)^{1/q} w_{q'}(x_{n+1} \dots x_m)$$

if we have picked s, t at the beginning so that these equalities hold $\forall$.

Taking inf over best possible representations yields the claim. $\blacksquare$

Lemma $\pi_p(T) = \nu_{p'}^*(T)$ (Apply definition)

— The π_2 - exception —

Prop Let E be a finite dimensional Banach space. Then $\nu_2(S\, E \to X) \leq \pi_2(S\, E \to X)$

(HW p-integral, show $\dim E < \infty \Rightarrow \nu_p \leq I_p$)

<u>Lemma</u> Let E be finite dimensional and $\varepsilon > 0$

Then there exist $m \in \mathbb{N}$ and

$\tilde{E} \subseteq \ell_\infty^m$ and $u : E \to \tilde{E}$ such that

$$(1-\varepsilon)\|x\| \leq \|u(x)\| \leq (1+\varepsilon)\|x\| \qquad \forall x \in E$$

(Notation: $\quad d_1(E, \tilde{E}) \leq \frac{1+\varepsilon}{1-\varepsilon}$)

<u>Proof</u>: A δ net $\subset B_{E^*}$ is a subset S

such that

$$\underset{s \neq t}{s, t \in S} \implies \|s-t\|_{E^*} > \delta$$

Since B_{E^*} is compact every δ-net has to be finite

Now take a maximal (finite) δ-net. S

Then for all

$$\boxed{x^* \in B_{E^*} \quad \exists s \in S \quad \|x^* - s\|_{E^*} \leq \delta}$$

Define $\quad u : E \to \ell_\infty(S) \quad u(x) = (s(x))_{s \in S}$

Clearly $\|u(x)\| \leq \|x\|$ for all $x \in E$.

Let $\|x\|_E = 1$. Then there exists $x^* \in B_{E^*}$ with $|x^*(x)| = 1$. Let $s \in S$ such that $\|s - x^*\|_{E^*} < \delta$. We get

$$\begin{aligned} |s(x)| &= |s(x) - x^*(x) + x^*(x)| \\ &\geq -|s(x) - x^*(x)| + 1 \\ &\geq 1 - \delta \underbrace{\|x\|}_{=} = 1 - \delta \end{aligned}$$

By homogeneity

$$(1-\delta)\|x\| \leq \sup_{s \in S} |s(x)| \leq \|x\| \qquad \forall x \in E$$

$\square$

Prop $\dim E < \infty$ $\quad \nu_2(T: E \to Y) \leq \pi_2(T)$

~~Lemma Let $E \subset F$ and $T: E \to Y$ be 2-summing. Then there exist $\hat{T}: F \to Y$ which is also 2-summing.~~

Proof

$$\begin{tikzcd} E \arrow[r] \arrow[rr, bend left, "T"] & E_2 \arrow[r, "\hat{T}"] & Y \end{tikzcd}$$

$$\begin{array}{c} \cap \\ C(B_{E^*}) \longrightarrow L_2(B_{E^*}, \mu) \end{array}$$

We have a factorization for T because

$$\|Tx\|_Y \leq \pi_2(T) \left(\int_{B_{E^*}} |x^*(x)|^2 \, d\mu \right)^{1/2}$$

and here

$$\|x\|_{E_2} = \left(\int_{B_{E'}} |x'(x)|^2 \, d\mu \right)^{1/2}$$

is the induced norm.

However, we want to change this

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, shift left] & Y \\ \tilde{E}_\varepsilon \subseteq \ell_\infty^m \arrow[u, shift left, "u_\varepsilon"] & \end{tikzcd}$$

Then $\quad Tu_\varepsilon^{-1} : \tilde{E}_\varepsilon \to Y \quad$ satisfies

$$\pi_2(Tu_\varepsilon^{-1}) \leq \pi_2(T) \, \|u_\varepsilon^{-1}\|$$

$\Longrightarrow$ (Grothendieck-Pietsch) $\quad \exists \; v_1 \dots v_m$ ~~on~~ $\sum v_j = 1$

$$\|Tu_\varepsilon^{-1}(x)\|_Y \leq \pi_2(T) \, \|u_\varepsilon^{-1}\| \left(\sum_{j=1}^m v_j \, |\langle x, s_j \rangle|^2 \right)^{1/2}$$

$$= \left(\sum_{j=1}^m |\langle x, \sqrt{v_j}\, s_j \rangle|^2 \right)^{1/2}$$

Let $\tilde{E}_2 \subseteq \ell_2^m$ be the subspace

given by $\quad E_2 = \{ \langle x, \sqrt{v_j}\, s_j \rangle \; ; \; x \in E \}$

Let $P \; \ell_2^m \to E_2$ be the orthonal projection

Then

$$\| \underbrace{T u_\varepsilon^{-1} P_2}_{S} : \hat{E}_2 \longrightarrow Y \| \le \|u_\varepsilon^{-1}\| \, \pi_2(T)$$

Moreover $x_j^* = s_j \sqrt{\nu_j}$ satisfy

$$\Big(\sum_{j=1}^{m} \|x_j^*\|_{E^*}^2\Big)^{1/2} \le \Big(\sum \|\nu_j\|\Big)^{1/2} \sup_j \|s_j\|_{E^*}$$

$$\le \|u_\varepsilon\|,$$

Hence we get

$$\boxed{\begin{aligned} & E \longrightarrow \ell_2^m \longrightarrow Y \\ & T = S\,w \\ & = \sum_{j=1}^{m} x_j^* \otimes S(e_j) \quad \in Y \end{aligned}}$$

$$\boxed{\begin{aligned} \nu_2(T) &\le \|u_\varepsilon^{-1}\| \, \|u_\varepsilon\| \, \pi_2(T) \\ &\le (1+\varepsilon)\tfrac{1}{1-\varepsilon} \, \pi_2(T) \end{aligned}}$$

□

Cor $\dim E < \infty$

$$|\mathrm{tr}(ST)| \le \pi_2(S)\, \pi_2(T)$$

Cor: $T\; X \to Y$ $\;S\; Y \to X$ one of them finite rank

then

i) $|tr(ST)| \le \pi_2(T)\, \pi_2(S)$

ii) $\pi_2(S) = \sup \{ |tr(ST)| \;|\; \pi_2(T) \le 1 ,\ T \text{ finite rank} \}$

Proof: a) T finite rank $X \xrightarrow{T} Y \xrightarrow{S} X$

$T(X) \subseteq E \subseteq Y$, $\quad S_E : E \to X$

Then $|tr(ST)| = |tr(S_E T)| = |tr(T S_E)|$

$$\le \pi_2(T)\, \nu_2(S_E)$$

$$\le \pi_2(T)\, \pi_2(S|_E)$$

$$\le \pi_2(T)\, \pi_2(S)$$

b) S finite rank: exercise

ii) i) shows $\ge$

$\le$ apply definit bc $\pi_2(S) \le \nu_2(S)$

$\square$

Lemma A: $u : \ell_2^n \to \ell_2^n$

$$\pi_2(u) = \Big(\sum_{ij} |\langle e_j, u(e_i) \rangle|^2 \Big)^{1/2}$$

Theorem (John) Let E be an n-dimensional Banach space. Then there exists a map $u : \ell_2^n \to E$ such that

$$\|u\| \, \pi_2(u^{-1}) = \sqrt{n}$$

($u(B_2^n) \subseteq B_E$ is the ellipsoid of maximal volume)

Proof. By Lewis lemma there exist $u : \ell_2^n \to E$ such that

$$\pi_2(u) \, \pi_2(u^{-1}) = n$$

We may assume $\pi_2(u) = \sqrt{n}$, $\pi_2(u^{-1}) = \sqrt{n}$.

By Pietsch factorization theorem we find

$$\begin{tikzcd} u \quad \ell_2^n \arrow[r, "v"] \arrow[d, hook] & E_2 \arrow[r, "w"] & E \arrow[r, "u^{-1}"] & \ell_2^n \\ C(K) \arrow[r] & L_2(\mu) \arrow[u] & & \end{tikzcd}$$

such that $\|w\| = 1$, $\pi_2(v) = \sqrt{n}$. Wlog we may assume $E_2 = \ell_2^n$. Then we have

$$\pi_2(v : \ell_2^n \to \ell_2^n) = \sqrt{n} \quad \pi_2(u^{-1}w : \ell_2^n \to \ell_2^n) = \sqrt{n}$$

and $\underbrace{(u^{-1}w)}_{a^{-1}} \underset{\substack{\| \\ a}}{v} = u^{-1} w v = u^{-1} u = Id$

Key observation

$$n = \operatorname{tr}(a^{-1}a) = \|a^{-1}\|_2 \|a\|_2 = \sqrt{n}\sqrt{n} \qquad (\circ)$$

$$\Downarrow$$

$a^{-1} = a^*$ hence a is unitary.

Indeed, we may use inner product on $\ell_2^{n^2}$

$$(b, a) = \sum_{ij} \bar{b}_{ij} a_{ij}$$

Then $(b, a) = \|b\|_2 \|a\|_2$ only holds for $b = \lambda a$ $\lambda \in \mathbb{K}$.

("Equality in Cauchy Schwarz)

$$\operatorname{tr}(b^* a) = \sum (b^*)_{jk} a_{kj} = \sum \bar{b}_{kj} a_{kj} = (b, a)$$

Thus $(\circ)$ is satisfied for $a = v$

Therefore we see that $u = wv$ satisfies

$\|u\| \leq 1$ $\pi_2(u^{-1}) \leq \sqrt{n}$ ✓

For the additional part we need another

Lemma B $\quad \pi_2(u) \leq \sqrt{n}\,\|u\| \quad \forall u: \ell_2^n \to X \quad$ i)

$\qquad\qquad \|\ \|^\vee = \nu_1^\circ(v) \leq \sqrt{n}\,\pi_2(v) \quad \forall v: X \to \ell_2^n \quad$ ii)

all we need $\nearrow$ $\qquad\qquad$ ($\dim X < \infty$)

Then we get

$\cancel{\pi_2(u)\,\pi_2(u^{-1}) \leq n}$

$$\|u\| \leq 1 \qquad \nu_1^\circ(u^{-1}) \leq \sqrt{n}\,\pi_2(u^{-1}) = n$$

Therefore $\qquad\qquad \alpha^* \ \text{for}\ \alpha = \|\cdot\|$

$$\|u\|\ \nu_1^\circ(u^{-1}) = n$$

By uniqueness of ellipsoid of maximal volume we deduce the assertion $\blacksquare$

Additional HW

Proof of lemma B via lemma A

i) $$\pi_2(u) = \pi_2(u\,\mathrm{Id}_{\ell_2^n}) \leq \|u\|\,\pi_2(\mathrm{Id}_{\ell_2^n})$$
$$= \|u\| \left(\sum_{i=1}^n \|e_i\|^2\right)^{1/2} = \|u\|\sqrt{n}$$

ii) let $v: X \to \ell_2^n \quad u: \ell_2^n \to X$

$$|\mathrm{tr}(vu)| \leq \pi_2(v)\,\pi_2(u) \leq \pi_2(v)\sqrt{n}\,\|u\|$$

Thus $\quad \|\ \|^\vee(v) \leq \sqrt{n}\,\pi_2(v)$. If in addition $\dim X < \infty$

then $\quad \|\ \|^\vee = \nu_1^\circ \quad$ ✗

Proof of (lemma A)

Let $a : \ell_2^n \to \ell_2^n$ Then

$$\left(\sum_{i=1}^{n} \|a(e_i)\|^2_{\ell_2^n}\right)^{1/2} \le \pi_2(a) \sup_{\|x^*\| \le 1} \left(\sum_{i=1}^{n} |\langle x^*(e_i)\rangle|^2\right)^{1/2}$$

$$\le \pi_2(a) \sup_{\|\alpha\|_2 \le 1} \left(\sum |\alpha_i|^2\right)^{1/2}$$

$$\le \pi_2(a)$$

$$\left(\sum_{i=1}^{n} \|a(e_i)\|^2_{\ell_2^n}\right)^{1/2} = \boxed{\|a\|_{HS}}$$

Thus $\|a\|_{HS} \le \pi_2(a)$

Observation $a : \ell_2^n \to X$ $\quad \left(\sum_{i=1}^{n} \|a(e_i)\|_X^2\right)^{1/2} \le \pi_2(a)$

Observation $b : X \to \ell_2^n$ $\quad \pi_2(b) \le \nu_2^0(b)$

$$\le \left(\sum_{i=1}^{n} \|b^*(e_i)\|^2\right)^{1/2}$$

Indeed $\quad X \xrightarrow{b^*(e_i)} \ell_2^n \xrightarrow{Id} \ell_2^n \quad w_2(e_i) \le 1$

Hence we get

$$\|a\|_{HS} \le \pi_2(a) \le \left(\sum_{i=1}^{n} \|a^*(e_i)\|^2_{\ell_2^n}\right)^{1/2} = \|a^*\|_{HS}$$

but $\|a\|_{HS} = \|a^*\|_{HS}$ and we are done ∎

Lemma Let X, Y be a real n-dimensional Banach space. and $T: X \to Y$ Then there are $x_1^* \ldots x_m^*$ $y_1 \ldots y_m$ such that

$$\upsilon_1^0(T) = \sum_{i=1}^{m} \|x_i^*\|_{X^*} \|y_i\|_Y \,, \quad T(x) = \sum_{i=1}^{m} x_i^*(x) y_i$$

and $m \leq$

Proof We consider the finite dimensional space $(V_1^0(X,Y), \upsilon_1^0)$. Then

$$\upsilon_1^0(T) = \inf \Big\{ \sum_{k=1}^{m} \|x_k^*\| \, \|y_k\| \; : \; x_k^* \in X^* \; y_k \in Y \Big\}$$
$$= \Big\{ \sum_{k=1}^{m} \lambda_k \|x_k^*\| \, \|y_k\| \quad |\textstyle\sum \lambda_k| \leq 1 \; \|x_k^*\| \leq 1 \; \|y_k\| \leq 1 \Big\}$$

Hence

$$\overline{Ext(B_{V_1^0})} \subseteq \{ x^* \otimes y : \|x^*\| = 1 \; \|y\| = 1 \}$$

the ~~e~~ extreme points (finite dimension, forget closure)

Thus $B_{V_1^0}$ is the convex hull of its extreme points Ext ⊆ Rank one tensors Dimension $V_1^0(X,Y)$ is n^2 Hence n^2+1 many ~~points~~ suffice.

Lemma Let $u: X \to Y$ $\dim X = n$ such that
$tr(uw) = 1$ $\nu(u) = 1$ $\|w\| = 1$

Then there $x_1^* \dots x_m^*$ $m \le n^2$ $u = \sum x_j^* \otimes y_j$
$y_1^* \dots y_m^*$ $m \le n^2$

and $\|w(y_j)\|_X = \|y_j\|_Y$
$\|w^*(x_j^*)\|_{Y^*} = \|x_j^*\|_{X^*}$

Proof $L = \{u: X \to Y : tr(uw) = 1\}$ is a n^2-1 dimensional ($\mathbb{R}$) affine hyperplane.

Claim $u \in \operatorname{conv}(L \cap \{x^* \otimes y : \|x^*\| = 1 = \|y\|\})$

Indeed $u = \sum_{j=1}^{m} x_j^* \otimes y_j$ $\sum \|x_j^*\| \|y_j\| = 1$

Then

$$1 = tr(wu) = \sum_j (x_j^*, w(y_j)) \le \sum \|x_j^*\| \|w(y_j)\|$$
$$\le \sum \|x_j^*\| \|w\| \|y_j\| \le 1$$

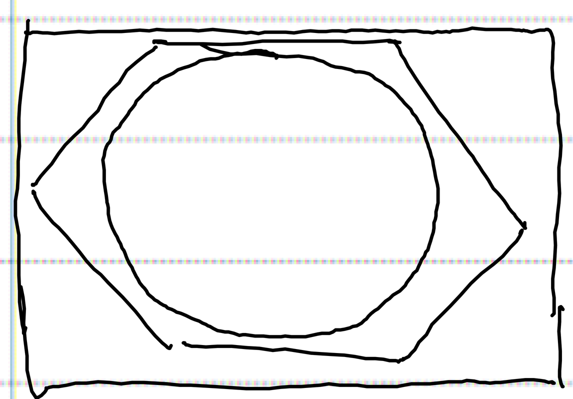
Thus we have equality and therefore $\|w(y_j)\| = \|y_j\|$
$\|w^*(x_j^*)\| = \|x_j^*\|$ and

$$(x_j^*, w(y_j)) = \|x_j^*\| \|w(y_j)\| = \|x_j^*\| \|y_j\| \quad \forall j$$

This means $\dfrac{x_j}{\|x_j^*\|} \otimes \dfrac{y_j}{\|y_j\|} \in L$! $\square$

→ Tomczak Jaegermann p=38 using selfadjoint element
(In Hilbert space this can be improved further!)

Comments



Hexagon — we need more than two vectors to write $Id = \sum x_j^* \otimes x_j$

I

II John's map $u = Id\ (\ell_2^n\ |\cdot|) \longrightarrow (\mathbb{R}^n\ \|\cdot\|_E)$

$$u^{-1} = \sum_{j=1}^{m} d_j\, x_j^* \otimes x_j \qquad |x_j^*| = \|x_j^*\| = 1$$

$$\|x_j\| = |x_j| = 1$$

Indeed use $w = u$ above and $v = \frac{u^{-1}}{n}$ than we get this ...

<u>Sauer Shelah lemma</u> Let $\dim E = n$. Therefore some subspace $F \subset E$ with $\dim F = \delta n$

$$\begin{array}{ccccc} \ell_2^n & \xrightarrow{u} & E & \xrightarrow{u^{-1}} & \ell_2^n \\ \cup & & & & \\ F & \longrightarrow & F & & \ell_2^n \end{array}$$

$$F \to \ell_\infty^n \xrightarrow{u} \ell_2^n \to \ell_2^n$$

One can find a factorization with n-points !

(Helpful tool).

Another interesting ellipsoid

$\mathbb{K} = \mathbb{R}$ let γ_n be the measure

$$d\gamma_n(x) = e^{-\frac{\|x\|^2}{2}} \frac{dx}{\sqrt{2\pi}^n}$$

Then the coordinate functions

$$g_k(x) = x_k$$

are independent normalized gaussian variables

$(N(0,1)$ $\mathbb{E}|g_k|^2 = 1$ $\mathbb{E} g_k = 0)$

Lemma The norm

$$\ell(u) = \left(\int_{\mathbb{R}^n} \| \sum_{k=1}^n g_k u(e_k) \|_X^2 \, d\mu \right)^{1/2}$$

is a right ideal norm for every real Banach space X.

Proof We have to show that $\ell(u\sigma) \leq \ell(u)$

for all contractions $\sigma : \ell_2^n \to \ell_2^n$.

Let us first assume that σ is a unitary

Then

$$\ell(u\sigma) = \left(\int_{\mathbb{R}^n} \Big\| \sum_{k=1}^{n} g_k \tilde{u}(e_k) \Big\|_X^2 \, d\mu(x)\right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^n} \| u\sigma(x) \|_X^2 \, d\mu(x)\right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^n} \| u\sigma(x) \|_X^2 \, e^{-\frac{|x|^2}{2}} \frac{dx}{\sqrt{2\pi}^n}\right)^{1/2}$$

$y = \sigma x$

$$= \left(\int_{\mathbb{R}^n} \| u(y) \|_X^2 \, e^{-\frac{|y|^2}{2}} \frac{dy}{\sqrt{2\pi}^n}\right)^{1/2} \qquad |y| = |x|$$

$$= \ell(u).$$

Now let w be an arbitrary matrix. We can find σ such that

$$w\sigma = d \qquad d \geq 0 \qquad \text{selfadjoint}$$

Let us diagonalize $d = v \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v^{-1}$ $\quad v$ orthogonal

Then $\|d\| \leq 1$ means $\sup_j |\lambda_j| \leq 1$

The extreme points of $B_{\ell_\infty^n}$ are the vectors (ε_j) $|\varepsilon_j| = 1$. Thus

$$\int |d\mu| = 1$$

$$w = d\sigma = v \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v^{-1} \sigma^{-1}$$

$$= \int v \begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{pmatrix} v^{-1} \sigma^{-1} \, d\mu(\varepsilon)$$

Thus

$$\ell(uw) = \int \ell(u \underbrace{v D_\varepsilon v^{-1}}_{\text{unitary}}) \, d\mu(\varepsilon)$$

$$= \ell(u). \qquad \boxed{}$$

Complex version: $H = \mathbb{C}^n = \mathbb{R}^{2n}$

$$g_k = \frac{g_k + i\, g_{k+n}}{\sqrt{2}} \qquad \text{complex gaussian.}$$

$$\ell(u) = \left(\mathbb{E} \left\| \sum_{k=1}^{n} g_k u(e_k) \right\|_X^2 \right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^{2n}} \left\| u\left(\frac{x+iy}{\sqrt{2}}\right) \right\|_X^2 e^{-\frac{\|x\|^2+|y|^2}{2}} \frac{dxy}{\sqrt{2\pi}^{\,2n}} \right)^{1/2}$$

Same argument applies, and now for complex unitaries.

$\boxed{\text{Theorem: The ellipsoid given by the } \ell\text{-norm is unique.}}$

Pisier's thm $\quad v: X \to \ell_2^n \quad \boxed{\ell(v^*) \le C(1+\log n)\, \ell^*(v)}$