0 Basic properties

Definition: A Banach space is given by a vector space X over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a norm $\|\cdot\| : X \longrightarrow [0,\infty)$ satisfying

i) $\|\lambda x\| = |\lambda| \, \|x\| \qquad \lambda \in \mathbb{K} \quad x \in X$

ii) $\|x+y\| \leq \|x\| + \|y\|$

iii) $\|x\| = 0 \iff x = 0$

iv) Every Cauchy sequence (x_n)
(this means $\forall \varepsilon > 0 \ \exists n_0 \ \forall n,m > n_0 \ \|x_n - x_m\| < \varepsilon$)
is convergent
(this means $\exists x \in X \ \forall \varepsilon > 0 \ \exists n_0 \ \forall n > n_0 \ \|x - x_n\| < \varepsilon$)

Remark: a) X satisfying i), ii), iii) is called normed space

b) Every normed space can be completed to satisfy iv)

c) iii) is equivalent to

$$\sum \|x_n\| < \infty \implies \sum x_n \in X$$