

II p -summing maps

$$\text{Def: } \omega_p(x_k) = \sup_{\|x^*\| \leq 1} \left(\sum_k |x^*(x_k)|^p \right)^{1/p}$$

$$u_{\vec{x}}(\alpha) = \sum \alpha_k x_k \quad \text{or} \quad u_{\vec{x}}(e_k) = x_k$$

$$\text{Lemma } \omega_p(x_k) = \|u_{\vec{x}}: \ell_{p'} \rightarrow X\|$$

$$\text{Proof } \|u_{\vec{x}}: \ell_{p'} \rightarrow X\| = \sup_{\sum |\alpha_k|^{p'} \leq 1} \|\sum \alpha_k x_k\|_X$$

$$= \sup_{\sum |\alpha_k|^{p'} \leq 1} \sup_{\|x^*\| \leq 1} \left| \sum \alpha_k x^*(x_k) \right|$$

$$= \sup_{\|x^*\| \leq 1} \left(\sum |x^*(x_k)|^p \right)^{1/p}$$

□

Q $u_{\vec{x}}$ well-defined? $p=1$ we need

$$u_{\vec{x}}: \mathbb{C} \rightarrow X$$

$$\{ \alpha_k \} : \lim_k \alpha_k = 0 \}$$

bc $\text{Span } \{e_k\}$ norm dense (unique extension by continuity)

Def $T: X \rightarrow Y$ is called p -summing if there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} \|Tx_k\|^p \leq C^p \sup_{\|x^*\| \leq 1} \sum_{k=1}^{\infty} |x^*(x_k)|^p \quad (*)$$

for all finite sequences $(x_k) \subset X$

$$\pi_p(T) = \inf \{ C \mid (*) \}$$

$$\pi_p(E, F) = \lambda T \{ \pi_p(T) \}$$

Note π_p is an ideal (H.W.)

Theorem $T: X \rightarrow Y$ is p -summing iff there exists a ^{Carleson} measure μ on B_{X^*} such that

$$\|Tx\| \leq C \left(\int |x^*(x)|^p d\mu \right)^{1/p} \quad (1 \leq p < \infty)$$

Moreover $C = \pi_p(T)$. (can be chosen)

Proof $\mathcal{C} = \{ f: (B_{X^*}, \sigma(X^*, X)) \rightarrow \mathbb{R} \mid \sup f < \infty \}$

\mathcal{C} is convex cone (i.e. convex and $r \geq 0, f \in \mathcal{C} \Rightarrow rf \in \mathcal{C}$)

$$\mathcal{C}_+ = \left\{ f_{\vec{x}}(x^k) = \sum_k \left(|x^k(x_k)|^p - \pi_p(\bar{1})^p \|Tx_k\|^p \right) \right\}$$

is also a cone. Indeed $\tilde{x}_k = \begin{cases} x_k & 1 \leq k \leq n \\ y_{k-n} & n < k \leq n+m \end{cases}$

$$f_{\vec{x}}(x^k) + f_{\vec{y}}(x^k) = \sum_{k=1}^{n+m} |x^k(x_k)|^p - \pi_p(\bar{1})^p \|Tx_k\|^p$$

$$\text{and } r f_{\vec{x}}(x^k) = f_{r\vec{x}}(x^k) \quad \forall x^k \quad \nabla$$

Moreover $\mathcal{C}_+ \cap \mathcal{C}_- = \emptyset$

By Hahn Banach there exists a real ^(continuous bc \mathcal{C}_+ -open) linear function $g: \mathcal{C}(\mathcal{B}_{\vec{x}^k}) \rightarrow \mathbb{R}$

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$$\beta = \sup_{g \in \mathcal{C}_-} \ell(g) \leq \inf_{g \in \mathcal{C}_+} \ell(g) = \alpha$$

Claim $\alpha \leq 0$

$$\text{If } \alpha > 0 \quad \alpha < \ell(g) < 2\alpha$$

$$\Rightarrow r\alpha < \ell(rg) < r2\alpha \xrightarrow{r \rightarrow 0} 0$$

$$\text{hence } \ell(rg) < \alpha \quad \text{for some } r \nabla$$

$$\beta \geq 0 \quad \text{same argument.}$$

$$\text{Thus } \beta = \alpha = 0$$

Claim $g \geq 0 \Rightarrow \ell(g) \geq 0$

Indeed $-g - \varepsilon 1 \in C$

$$\left. \begin{array}{l} \ell(-g - \varepsilon 1) \leq 0 \\ \ell(g) + \varepsilon \ell(1) \geq 0 \end{array} \right\} \Rightarrow \ell(g) \geq 0$$

By Riesz Representation theorem, there exists a positive measure μ on $C(\mathbb{B}_{\mathbb{R}^n})$ such that

$$\int g d\mu \geq 0 \quad \forall g \in C_+$$

$$\text{Thus } \int g \frac{d\mu}{\mu(\mathbb{B}_{\mathbb{R}^n})} \geq 0 \quad \forall g \in C_+$$

Hence we may assume $\mu(\mathbb{B}_{\mathbb{R}^n}) = 1$

Therefore we get

$$\int |x^k(\nu)|^p - \pi_p(\Gamma)^p \|Tx\|^p d\mu \geq 0$$

$$\Rightarrow \int |x^k(\nu)|^p \geq \pi_p(\Gamma)^p \|Tx\|^p$$

Conversely, let $x_1, \dots, x_n \in X$ then

$$\sum \|Tx_n\|^p \leq C^p \int \sum |x^k(x_n)|^p d\mu \leq C^p \sup_{\|x^k\| \leq 1} \sum |x^k(x_n)|^p$$

Remark: let $S \subset B_{X^*}$ be a closed subset such that still

$$\sup_{\lambda \in S} |\lambda^*(x)| = \|x\|$$

Then we may replace B_{X^*} by S
(bc we don't change ω_p -norm)

Example / Cor let $T: C_0 \rightarrow X$ be p -summing then there exists $\mu_n \geq \sum \mu_n = 1$ such that

$$\|Tx\|_X^p \leq \sum_k \mu_k \|x_k\|_p^p$$

Indeed, we may use $S = \mathcal{NT}$ with the ~~norm~~
 $\sigma(B_1, C_0)$ topology

$$\left(\begin{array}{c} C_0 \subset C(B_{B_1}) \\ \cup \\ \subseteq C_0(\mathbb{N}) \end{array} \right)$$

$K = \overline{\mathcal{NT}}$ one point compactification \uparrow

Gen rem

$$\left(\begin{array}{c} \sum_{n \in \mathbb{N}} T_n y \\ \subseteq C(\mathbb{N}) \end{array} \right)$$

\Rightarrow Frank

$$\|Tx\|_p \leq \sum_k \mu_k \|x_k\|_p^p$$

which we may ignore later

trace duality in this context

Recall $X \rightarrow Y$ Banach spaces & norm

$$\alpha^*(S) = \sup \{ \text{tr}(TS) \mid S \text{ finite rank } \alpha(S) \leq 1 \}$$

is called the adjoint norm

(Rem: there are variations...)

Here $S: Y \rightarrow X$ finite rank, f

$$S(y) = \sum_{j=1}^n y_j^*(y) x_j$$

$$\text{then } TS(y) = \sum y_j^*(y) T x_j$$

and we define

$$\text{tr}(TS) = \sum_{j=1}^n y_j^*(T x_j)$$

well-defined?

Lemma $T: Y \rightarrow Y$ finite rank

$$\left. \begin{aligned} T(y) &= \sum y_j^*(y) y_j \\ T(y) &= \sum \widehat{y}_k^*(y) \widehat{y}_k \end{aligned} \right\} = \sum y_j^*(y) = \sum \widehat{y}_k^*(\widehat{y}_k)$$

Proof let $E = T(Y) \subseteq Y$ be the range of T

let $E \subset F$ such that $y \in F, \tilde{y}_k \in F$

let z_1, \dots, z_m be a basis of F
 z_1^*, \dots, z_m^* dual functionals

Claim $T(y) = \sum_{j=1}^m z_j^*(T(y)) z_j$ obvious

Claim $\sum z_j^*(T z_j) = \sum y_j^*(y_j)$

Indeed

$$\begin{aligned} T(y) &= \sum_{j=1}^m y_j^*(y) y_j \\ &= \sum_{j=1}^m y_j^*(y) \sum_{k=1}^m z_k^*(y_j) z_k \\ &= \sum_{k=1}^m \left(\sum_j z_k^*(y_j) y_j^*(y) \right) z_k \end{aligned}$$

This means $z_k^*(T(y)) = \sum_j y_j^*(y) z_k^*(y_j)$ by !!

Therefore

$$\begin{aligned} \sum_k z_k^*(T(z_k)) &= \sum_k \sum_j y_j^*(z_k) z_k^*(y_j) \\ &= \sum_j \sum_k y_j^*(z_k) z_k^*(y_j) \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^n y_j^v(y_j) &= \sum_{j=1}^n y_j^v \left(\sum_k z_k^v(y_j) z_k \right) \\ &= \sum_{j=1}^n z_k^v(y_j) y_j^v(z_k) = \sum_k T^v z_k^v(z_k) \end{aligned}$$

The same holds for the \sim and we are done. \square

Pb find α such that

$$\boxed{\alpha^*(s) = \mathbb{T}_p(s)}$$

Motivation:

$$\left(\sum_1^n \|Tx_k\|_Y^p \right)^{1/p} = \sup_{\left\{ \sum \|z_k\|_{Y'}^{p'} \leq 1 \right\}} \left| \sum \langle y_k^v, Tx_k \rangle \right|$$

$$\begin{array}{ccc} T: X & \longrightarrow & Y \\ \uparrow & & \downarrow \\ u_{\vec{x}} \hat{=} (x_k) & & y \\ & & \downarrow \\ & & \ell_{p'}^m = \ell_{p'}^m \quad (y_k^v(y))_{k=1}^n \end{array}$$

$$\text{Def } \omega_q^0(S: Y \rightarrow X) = \inf_{\vec{y}} \omega_q^1(x_k) \left(\sum \|x_k^v\|_{Y'}^q \right)^{1/q}$$

$$S(y) = \sum_{k=1}^n y_k^v(y) x_k$$

Lemma ω_q^0 defines a norm on $\mathcal{F}(Y, X) =$ finite rank maps

Proof

We need $ab = \inf_{t>0} \frac{(ta)^q}{q} + \frac{(t'b)^{q'}}{q'}$

$$S_1 = \sum_{k=1}^n y_k^* \otimes x_k = \sum_{k=1}^n s y_k^* \otimes s^{-1} x_k$$

$$S_2 = \sum_{j=n+1}^m y_j^* \otimes x_j = \sum_{j=n+1}^m t y_j^* \otimes t^{-1} x_j$$

\uparrow
 convenience

when $|y^* \otimes x(y)| = y^*(x) |x|$

is the associated linear map

let $t > 0$ be arbitrary
 $s > 0$

① $t = s = 1$

$$S_1 + S_2(y) = \sum_{j=1}^m y_j^* \otimes y_j$$

$$V_q^0(S_1 + S_2) \leq \left(\sum_{j=1}^m \|y_j^*\|_q \right)^{1/q} W_{q'}(x_1, \dots, x_m)$$

~~$$\leq \left(\sum_{j=1}^m \|y_j^*\|_q \right)^{1/q} + \left(\sum_{j=n+1}^m \|y_j^*\|_q \right)^{1/q}$$~~

$$\leq \frac{1}{q} \sum_{j=1}^m \|y_j^*\|_q + \frac{1}{q'} \sup_{x^*} \sum_{j=1}^m |x^*(y_j)|^{q'}$$

$$\leq \frac{1}{q} \sum_{j=1}^n \|y_j^*\|_q + \frac{1}{q} \sum_{j=n+1}^m \|y_j^*\|_q + \frac{1}{q'} \sup_{x^*} \sum_{j=1}^n |x^*(y_j)|^{q'} + \frac{1}{q'} \sup_{x^*}$$

$$= \left[\frac{1}{q} \sum_{j=1}^n \|y_j\|^q + \frac{1}{q'} \sup_{x \sim} \sum_{j=1}^n |x^*(y_j)|^{q'} \right] + \left[\right]$$

$$= \left(\sum_{j=1}^n \|y_j\|^q \right)^{1/q} \omega_{q'}(x, \{y_j\}) + \left(\sum_{j=1}^n \|y_j\|^q \right)^{1/q} \omega_{q'}(x_{ur}, \{y_j\})$$

if we have picked s, t at the beginning so that these equalities hold ∇

Taking up over best possible representations yields the claim. \square

Lemma $\Pi_p(T) = \mathcal{U}_p^*(T)$ (Apply definition)

— The Π_2 -exception —

Prop let E be a finite dimensional Banach space
 Then $\mathcal{U}_2(S: E \rightarrow X) \leq \Pi_2(S: E \rightarrow X)$

(HW p -integral, show $\dim E < \infty \Rightarrow \mathcal{U}_p \leq I_p$)