NONCOMMUTATIVE BURKHOLDER/ROSENTHAL INEQUALITIES II: APPLICATIONS

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Abstract. We show norm estimates for the sum of independent random variables in noncommutative L_p spaces for $1<p<\infty$ following previous work by the authors. These estimates generalize Rosenthal's inequalities in the commutative case. Among other applications, we derive a formula for p-norm of the eigenvalues for matrices with independent entries, and characterize those symmetric subspaces and unitary ideal spaces which can be realized as subspaces of noncommutative L_p for $2 < p < \infty$.

0. Introduction and Notation

Martingale inequalities have a long tradition in probability. The applications of the work of Burkholder and his collaborators [B73, ?, BDG72, B71a, B71b, BGS71, BG70, B66] ranges from classical harmonic analysis to stochastical differential equations and the geometry of Banach spaces. When proving the estimates for the 'little square function' Burkholder was aware of Rosenthal's result [Ros] on sums of independent random variables. Here we proceed differently and prove the noncommutative Rosenthal inequality along the same line as the noncommutative Burkholder inequality from [JX1]. This slight modified prove yields a better constant. The main intention of this paper is to illustrate the usefulness of the 'little square function' in several examples. For many applications it is important to consider generalized notations of independence. This will allow us to explore applications towards random matrices and symmetric subspaces of noncommutative L_p spaces.

Our estimates on random matrices are motivated by the noncommutative Khintchine inequality. In [LP] Lust-Piquard showed that for $2 \le p < \infty$ and scalar coefficients (a_{ij}) one has

$$(0.1) \qquad \mathbb{E}\|\sum_{ij} \varepsilon_{ij} a_{ij} e_{ij}\|_p \sim_{c(p)} \left(\sum_i \left(\sum_j |a_{ij}|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} + \left(\sum_j \left(\sum_i |a_{ij}|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} .$$

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We use the notation $a \sim_c b$ if $a \le c_1 b$, $b \le c_2 a$ and $c_1c_2 \le c$. Let N be a von Neumann algebra with a normal faithful trace τ. Then the L_p-norm of an operator x affiliated to N is given by

$$\|x\|_p \;=\; [\tau(|x|^p)]^{\frac{1}{p}} \,.$$

In particular for $N = B(\ell_2)$ and $\tau = tr$, the p-norm of a matrix is the p-norm of its singular values

$$\|\sum_{ij} a_{ij}e_{ij}\|_p \;=\; \left(\sum_k \lambda_k(|a|)^p\right)^{\frac{1}{p}} \,,$$

i.e. the eigenvalues $\lambda_k(|a|)$ of $|a|$. In our first result we replace the coefficients $a_{ij}\varepsilon_{ij}$ in (0.1) by arbitrary random variables:

Theorem 0.1. *Let $(f_{ij}) \subset L_p(\Omega,\mu)$ be a matrix of independent mean 0 random variables defined on a probability space.*

i) *If $2 \le p < \infty$, then*

$$\|\sum_{ij} f_{ij}e_{ij}\|_p$$

$$\sim_{C_p} \max\left\{\left(\sum_{ij}\|f_{ij}\|_p^p\right)^{\frac{1}{p}}, \left(\sum_j\left(\sum_i\|f_{ij}\|_2^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_i\left(\sum_j\|f_{ij}\|_2^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\} .$$

ii) *If $1 < p \le 2$, then*

$$\|\sum_{ij} e_{ij}\otimes f_{ij}\|_p \sim_{C_{p'}} \inf_{f_{ij}=g_{ij}+h_{ij}+d_{ij}}$$

$$\left(\sum_j\|\left(\sum_i \mathbb{E}(g_{ij}^*g_{ij})\right)^{\frac{1}{2}}\|_p^p\right)^{\frac{1}{p}} + \left(\sum_i\|\left(\sum_j \mathbb{E}(h_{ij}h_{ij}^*)\right)^{\frac{1}{2}}\|_p^p\right)^{\frac{1}{p}} + \left(\sum_{ij}\|d_{ij}\|_p^p\right)^{\frac{1}{p}} .$$

Here the infimum is taken of g_{ij}, h_{ij} d_{ij} with mean 0 and measurable with respect to σ-algebra generated by f_{ij}.

The estimates in Theorem 0.1 for $p \ge 2$ is a direct application of our main result

$$\|\sum_k x_k\|_p \sim_{cp} \big(\sum_k \|x_k\|_p^p\big)^{\frac{1}{p}} + \|\big(\sum_k E(x_k^*x_k + x_kx_k^*)\big)^{1/2}\|_p \tag{0.2}$$

which holds for independent mean 0 variables. Here E is allowed to be operator valued. This allows us to replace the f_{ij}'s by operator valued (or matrix valued) coefficients provided they satisfy appropriate independence conditions, for example if they are freely independent (in the sense of Voiculescu [VDN]). We also extend our result to the non-tracial setting and even non-faithful setting. The dual version of Rosenthal's inequality (in the non-tracial setting) provides Khintchine type inequalities for the generators of the CAR algebra. As in the case of Khintchine inequalities [LPP] one has to replace the maximum by an infimum, see section 4. In section 3 we provide a version of Rosenthal's inequality using the recent concept of maximal functions in the noncommutative setting. Formally these formulations are much closer to Burkholder's original version [B73] and we are providing a stopping time free proof of these results for $p > 2$.

Symmetric subspaces of L_p spaces are motivated by probabilistic notions of exchangeable random variables. In the commutative situation. The memoir of Johnson, Maurey, Schechtman and Tzafriri [JMST] contains an impressive amount of information and many sophisticated applications of probabilistic techniques. In this paper we extend one of the more elementary inequalities from [JMST]:

Theorem 0.2. *Let N and M be von Neumann algebras and $2 \le p < \infty$. Let $e_k \in L_p(N)$ and $C > 0$ such that*

$$\|\sum_k \varepsilon_k a_{\pi(k)} \otimes x_k\|_p \ \le\ C\|\sum_k a_k \otimes x_k\|_p \tag{0.3}$$

holds for all $\varepsilon_k = \pm 1$, all permutations π on $\{1,..,n\}$ and coefficients $a_k \in L_p(M)$. Then there are constants a, c, r such that

$$\|\sum_k a_k \otimes e_k\|_p \sim_{C^2 c_p} a\left(\sum_{k=1}^n \|a_k\|_p^p\right)^{\frac{1}{p}} + b\|(\sum_{k=1}^n a_k^* a_k)^{\frac12}\|_p + c\|(\sum_{k=1}^n a_k a_k^*)^{\frac12}\|_p\,. \tag{0.4}$$

holds for all $a_k \in L_p(M)$.

In Banach space theory a sequence (e_k) satisfying (0.3) for scalar valued coefficients is called a symmetric basis for $\overline{\mathrm{span}}\{e_k : k \in \mathbb{N}\}$. In analogy with the commutative case, we obtain a characterization of symmetric subspaces for $2 < p < \infty$.

Corollary 0.3. *Let $2 \le p < \infty$ and $X \subset L_p(N)$ be a subspace with a symmetric basis, then X is isomorphic to ℓ_p or to ℓ_2.*

We also explore the natural operator space version (operator-valued version) of this result. A further application is a characterization of unitary ideals (and noncommutative functions spaces) in noncommutative L_p spaces for $p > 2$. The non-faithful version of Rosenthal's allows us to show that L_p spaces are asymptotically symmetric (see [JR]). The results in [JR] show that Corollary 0.3 still holds for spaces with a subsymmetric basis (see section 5 for a precise definition).

The paper is organized as follows. Section 1 contains basic information on noncommutative L_p spaces and the abstract version of the noncommutative Rosenthal inequality for $2 \leq p < \infty$. In section 2 we extend these results to the non-faithful situation. Section 3 develops an improvement of the noncommutative Rosenthal inequality using the recent theory of maximal functions and vector-valued L_p spaces. In section 4 we investigate the case $1 < p \leq 2$. We refer to section 5 for more information on symmetric subspace of noncommutative L_p for $p \geq 2$. Applications to unitary ideals are contained in section 6.

We use standard notation from (noncommutative) probability and the theory of von Neumann algebras (see e.g. [KRI, KRII], [SZ, Str] and [TakI, TakII, TakIII]). For more information and definitions on operator space theory we refer to [Pis2] and [ER]. We refer to [Pis1, JNRX] for the natural operator space structure on L_p spaces.

1. Rosenthal's inequality for sums of independent random variables

In this section, we shall prove a noncommutative version of Rosenthal's inequality. We will work under the standard assumptions from [JX1], i.e. we fix a faithful normal state ϕ on a von Neummann algebra N and we denote by D the density of ϕ in the Haagerup's space $L_1(N)$ such that

$$\phi(x) \;=\; tr(xD)\,.$$

We refer to [JX1] for details on the Haagerup L_p-spaces needed in this paper. In our situation, we may work with $N \rtimes_{\sigma_t^\phi} \mathbb{R}$, where σ_t^ϕ is the modular group associated to ϕ. $N \rtimes \mathbb{R}$ is semifinite and posseses a unique trace τ such that $\tau(\theta_s(x)) \;=\; e^{-s}\tau(x)$ is satisfied for the dual action. Then $T(x) \;=\; \int_{\mathbb{R}} \theta_s(x)ds$ is a positive operator valued weight and D is the density of the weight $\phi \circ T$ in the space $L_0(N \rtimes \mathbb{R}, \tau)$ of measurable operators affiliated to $N \rtimes \mathbb{R}$. Let us recall that

$$L_p(N) \;=\; \{x \in L_0(N,\tau) \;:\; \theta_s(x) \;=\; e^{-\frac{s}{p}}x\}\,.$$

It turns out that for $x \in L_p(N)$ with polar decomposition $x = u|x|$, we have $u \in N$ and $|x| \in L_p(N)$. The norm in $L_p(N)$ is given by

$$\|x\|_p \;=\; (tr(|x|^p))^{\frac{1}{p}}\,.$$

Here $tr \,:\, L_1(N) \to \mathbb{C}$, $tr(D_\phi) \;=\; \phi(1)$ is the Haagerup trace. We will frequently use Hölder's inequality

$$\|xy\|_p \;\leq\; \|x\|_r\|y\|_q$$

whenever $\frac{1}{p} = \frac{1}{r} + \frac{1}{q}$. We say that $M \subset N$ is a *ϕ-invariant subalgebra*, if $\sigma_t^\phi(M) \subset M$. According to Takesaki's Theorem ([Tak72]), there exists a unique conditional expectation

$E : N \to M$ such that $\phi|_M \circ E = \phi$. This implies in particular

$$\sigma_t^{\phi} \circ E \;=\; E \circ \sigma_t^{\phi} \tag{1.1}$$

for all $t \in \mathbb{R}$. For $1 \le p \le \infty$, the conditional expectation E extends to a contraction $E_p : L_p(N) \to L_p(M)$ densely defined as follows

$$E_p(xD_{\phi}^{\frac{1}{p}}) \;=\; E(x)D_{\phi|_M}^{\frac{1}{p}} \,.$$

Here $D_{\phi|_M}$ is the density of $\phi|_M$ in $L_1(M)$. However, equation (1.1) implies that $M \rtimes_{\sigma_t^{\phi}}^{\phi|_M} \mathbb{R} \subset N \rtimes_{\sigma_t^{\phi}} \mathbb{R}$ and that the restriction of τ to $M \rtimes \mathbb{R}$ is the unique trace for $M \rtimes \mathbb{R}$. We obtain an isometric embedding $\iota_p : L_p(M) \to L_p(N)$ such that $\iota_p(D_{\phi|_M}^{1/p}) = D_{\phi}^{1/p}$ (see [JX1] for more details). Therefore, we will simple use D for the density of ϕ. Let us recall that the conditional expectation E is characterized by

$$\phi(E(x)y) \;=\; \phi(xy) \tag{1.2}$$

for all $x \in N$ and $y \in M$. If we have several ϕ-invariant subalgebras (M_i), we write (E_{M_i}) for the corresponding sequence of conditional expectations. We refer to [JX1] for the following frequently used equality

$$E_p(D^{\frac{1-\theta}{p}}xD^{\frac{\theta}{p}}) \;=\; D^{\frac{1-\theta}{p}}E(x)D^{\frac{\theta}{p}} \tag{1.3}$$

which holds for all $x \in N$, all $0 \le \theta \le 1$ and $1 \le p \le \infty$. It is convenient to omit the index p. This is justified using Kosaki's embedding $I : L_p(N) \to L_1(N)$, $I(xD^{\frac{1}{p}}) \;=\; xD$ since then $E_1(I(y)) \;=\; I(E_p(y))$. In this sense all the maps E_p are 'induced' by the same map E_1.

For our formulation of Rosenthal's inequality (and its natural matricial valued generalizations) we will assume that $M \subset N$ is ϕ- invariant and that there are further ϕ-invariant von Neumann subalgebras $(A_k) \subset N$ *containing* M. We will say that (A_k) are (faithfully) *independent over* M if

I) $E_M(xy) = E_M(x)E_M(y)$ holds for all $x \in A_k$ and y in the algebra generated by $A_1, ..., A_{k-1}, A_{k+1}, ..., A_n$.

The following notion has been introduced by Schürmann: The algebras (A_k) are *increasingly independent* over M if

II) $E_{A_1\cup\cdots\cup A_{k-1}}(x) \;=\; E_M(x)$ holds for all $x \in A_k$ and $1 \le k \le n$

Lemma 1.1. *Condition* I) *implies condition* II). *Moreover, if* I) *is satisfied and* $x_k \in L_p(A_k)$ *satisfy* $E_M(x_k) = 0$, *then*

$$\|\sum_{k=1}^{n} \varepsilon_k x_k\|_p \;\le\; 2\|\sum_{k=1}^{n} x_k\|_p$$

holds for all $\varepsilon_k = \pm 1$ *and* $1 \le p \le \infty$.

Proof. Let us assume that condition I) is satisfied and let $S \subset \{1, ..., n\}$ be a subset of $\{1, ..., n\}$ such that $k \notin S$. Since all the A_k are ϕ-invariant so is the von Neumann subalgebra B_S generated by the $\{A_k | k \in S\}$. Let us denote by $\mathcal{E}_S$ the unique conditional expectation characterized by

$$\phi(\mathcal{E}_S(x)y) \;=\; \phi(xy)$$

for all $y = a_1 \cdots a_m$, $a_j \in A_{i_j}$, $i_j \in S$. Now, we assume that $x \in A_k$ and $E_M(x) = 0$. Let $y = a_1 \cdots a_m$ as above. Then we deduce from condition I) that

$$\phi(xy) \;=\; \phi(E_M(xy)) \;=\; \phi(E_M(x)E_M(y)) \;=\; 0\,.$$

Thus $\mathcal{E}_S(x) = 0$. Hence for an arbitrary element $x \in A_k$, we get

$$\mathcal{E}_S(x) \;=\; \mathcal{E}_S((x - E_M(x)) + \mathcal{E}_S(E_M(x)) \;=\; E_M(x)$$

because $M \subset B_S$. If we apply this to the set $S = \{1, ..., k-1\}$ we obtain condition II). To prove the second assertion consider $\varepsilon_k = \pm 1$ and define $S = \{k : \varepsilon_k = 1\}$. By approximation with elements of the form $x_k = a_k D^{\frac{1}{p}}$, $a_k \in A_k$ and $E_M(a_k) = 0$, we see that

$$\mathcal{E}_S(\sum_{k=1}^{n} x_k) \;=\; \sum_{k\in S} x_k + \sum_{k\notin S} \mathcal{E}_S(x_k) \;=\; \sum_{k\in S} x_k\,.$$

Since $\mathcal{E}_S$ is a contraction, we deduce

$$\|\sum_{k=1}^{n} \varepsilon_k x_k\|_p \le \|\sum_{k\in S} x_k\|_p + \|\sum_{k\in S^c} x_k\|_p \;=\; \|\mathcal{E}_S(\sum_{k=1}^{n} x_k)\|_p + \|\mathcal{E}_{S^c}(\sum_{k=1}^{n} x_k)\|_p \;\le\; 2\|\sum_{k=1}^{n} x_k\|_p\,. \blacksquare$$

Let us formulate our version of Rosenthal's inequality. Following [JX1], we introduce the notations

$$\|(x_k)\|_{\ell_p(L_p)} \;=\; \left(\sum_k \|x_k\|_p^p\right)^{\frac{1}{p}}$$

$$\|(x_k)\|_{L_p(M,E;\ell_2^c)} \;=\; \|\sum_k E(x_k^* x_k)\|_{p/2}^{\frac{1}{2}} \quad \text{and} \quad \|(x_k)\|_{L_p(M,E;\ell_2^r)} \;=\; \|\sum_k E(x_k x_k^*)\|_{p/2}^{\frac{1}{2}}\,.$$

Here E is a conditional expectation. For many applications it turns out to be quite useful to work with these simplified 'little square functions'.

Theorem 1.2. *Let $2 \le p < \infty$, $n \in \mathbb{N} \cup \{\infty\}$. Let $M \subset N$, $(A_k) \subset N$ von Neumann subalgebras and $x_k \in L_p(A_k)$ with $E_M(x_k) = 0$.*

I) *If the $(A_k)_{1 \le k \le n}$ are independent over M, then*

$$\frac{1}{Cp} \| \sum_{k=1}^{n} x_k \|_p \le \max\{ \|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(M,E;\ell_2^c)}, \|(x_k)\|_{L_p(M,E;\ell_2^r)} \} \ \le \ 2 \| \sum_{k=1}^{n} x_k \|_p \, .$$

II) *If the $(A_k)_{1 \le k \le n}$ are increasingly independent over M, then*

$$\frac{1}{Cp^2} \| \sum_{k=1}^{n} x_k \|_p \le \max\{ \|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(M,E;\ell_2^c)}, \|(x_k)\|_{L_p(M,E;\ell_2^r)} \} \ \le \ 2 \| \sum_{k=1}^{n} x_k \|_p \, .$$

Here C is an absolute constant.

Proof. Since the algebra N_n generated by $A_1, ..., A_n$ is ϕ-invariant, we may assume that $N = N_n$. The space $L_p(N) = (1 - E_M) L_p(N)$ is 2-complemented in $L_p(N)$ and therefore the lower estimate

$$\left(\sum_{k=1}^{n} \|x_k\|_p \right)^{\frac{1}{p}} \ \le \ 2 \| \sum_{k=1}^{n} x_k \|_p$$

follows by interpolation (using the inclusion map $x \to x D^{\frac{1}{2}} \in L_2(N)$, see [Kos] for more details on these interpolation spaces). Moreover, for $x = \sum_{k=1}^{n} x_k$, we have

$$\| \sum_{k=1}^{n} E_M(x_k^* x_k) \|_{\frac{p}{2}} = \|E_M(x^* x)\|_{\frac{p}{2}} \ \le \ \|x^* x\|_{\frac{p}{2}} \ = \ \|x\|_p^2 \, .$$

Therefore the lower estimates for the norm of the sum are proved. The main part is the proof of the upper estimate. We will now consider the case I). Let us note that for $1 \le p \le 2$, we deduce from interpolation (and the fact that the space of mean 0 is 2-complemented in $L_p(A_k)$ see above) that $E_M(x_k) = 0$ implies

$$\| \sum_{k=1}^{n} x_k \|_p \ \le \ 2 \left(\sum_{k=1}^{n} \|x_k\|_p^p \right)^{\frac{1}{p}} \, .$$

We will use the standard iteration procedure. Indeed, we may assume that for $q = \frac{p}{2}$, we have the estimate

$$\| \sum_{k=1}^{n} x_k \|_q \ \le \ C(q) \max\{ \|(x_k)\|_{\ell_q(L_q)}, \|(x_k)\|_{L_q(M,E;\ell_2^c)}, \|(x_k)\|_{L_q(M,E;\ell_2^r)} \}$$

for all $x_k \in L_q(A_k)$ with mean 0. Our aim is to prove the estimate for p. Of course we may assume $p > 2$. Let $x_k \in L_p(A_k)$ and $E_M(x_k) = 0$. First we apply the Khintchine inequality (see [PS] for the right order of constants) and deduce from Lemma 1.1 that

$$\|\sum_{k=1}^n x_k\|_p \;\le\; 2\,\mathbb{E}\|\sum_{k=1}^n \varepsilon_k x_k\|_p \;\le\; 2c_1\sqrt{p}\,\max\{\|\sum_{k=1}^n x_k^*x_k\|_{\frac{p}{2}}^{\frac12}, \|\sum_{k=1}^n x_kx_k^*\|_{\frac{p}{2}}^{\frac12}\}\,. \tag{1.4}$$

Let us consider the first one of these two square functions. We define the selfadjoint mean 0 elements $y_k = x_k^*x_k - E_M(x_k^*x_k)$. By hypotheses we have

$$\begin{aligned}\|\sum_{k=1}^n x_k^*x_k\|_{\frac{p}{2}} &\le \|\sum_{k=1}^n E_M(x_k^*x_k)\|_{\frac{p}{2}} + \|\sum_{k=1}^n y_k\|_{\frac{p}{2}}\\ &\le \|\sum_{k=1}^n E_M(x_k^*x_k)\|_{\frac{p}{2}} + C(q)\max\{\|(y_k)\|_{\ell_q(L_q)}, \|(y_k)\|_{L_q(M,E;\ell_2^c)}, \|(y_k)\|_{L_q(M,E;\ell_2^r)}\}\end{aligned}$$

Moreover, if $2 \le p \le 4$, we have $1 \le q = p/2 \le 2$ and we can disregard the second term. Since E_M is a contraction, we have

$$\left(\sum_{k=1}^n \|y_k\|_q^q\right)^{\frac1q} = \left(\sum_{k=1}^n \|x_k^*x_k - E_M(x_k^*x_k)\|_q^q\right)^{\frac1q} \le 2\left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac2q}\,.$$

Hence, for $2 \le p \le 4$, we find

$$\|\sum_{k=1}^n x_k\|_p \le 2c_1\sqrt{p}(1+4)^{\frac12}\max\{\|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(M,E;\ell_2^c)}, \|(x_k)\|_{L_p(M,E;\ell_2^r)}\}\,.$$

For $4 < p < \infty$, we first note that

$$\begin{aligned}E_M(y_k^*y_k) &= E_M((x_k^*x_k - E_M(x_k^*x_k))^*(x_k^*x_k - E_M(x_k^*x_k)))\\ &= E_M(x_k^*x_kx_k^*x_k) - E_M(x_k^*x_k)E_M(x_k^*x_k) \;\le\; E_M(|x_k|^4)\,.\end{aligned}$$

Now, we may use the interpolation result of [JX1, Lemma 5.2] (for $\mathcal{E}_{n-1} = E_M$) and find

$$\|\sum_{k=1}^n E_M(|x_k|^4)\|_{\frac{q}{2}} \le \|\sum_{k=1}^n E_M(|x_k|^2)\|_q^{\frac{q-2}{q-1}}\left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac{1}{q-1}}\,.$$

By homogeneity this implies

$$\|\sum_{k=1}^n E_M(|x_k|^4)\|_{\frac{q}{2}}^{\frac12} \le \max\left\{\left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac2p}, \|\sum_{k=1}^n E_M(|x_k|^2)\|_q\right\}\,.$$

Therefore we have proved that

$$\|(y_k)\|_{L_q(M,E;\ell_2^c)} \le \max\{\|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(M,E;\ell_2^c)}\}^2\,.$$

Applying the same argument for $x_k x_k^*$, we find

$$\|\sum_{k=1}^n x_k\|_p \le 2c_1\sqrt{p}(1+2C(q))^{\frac{1}{2}} \max\{\|(x_k)\|_{\ell_p(L_p)}, \|(x_k)\|_{L_p(M,E;\ell_2^c)}, \|(x_k)\|_{L_p(M,E;\ell_2^r)}\} \,.$$

This shows $C(p) \le 2c_1\sqrt{p}(1+2C(q))^{\frac{1}{2}}$ for $p > 4$. We have seen that for $2 \le p \le 4$, we may assume $C(q) \le C_1 q$. Thus by induction hypothesis we may assume $C(q) \le Cq$ for some constant $C \ge C_1 \ge 1$ and find

$$C(p) \;\le\; 2c_1\sqrt{p}(1+2Cp/2)^{\frac{1}{2}} \;\le\; 2\sqrt{2}c_1 C^{\frac{1}{2}} p \,.$$

Thus for $C = \max\{C_1, 8c_1^2\}$ the induction argument works and we obtain the assertion I). For increasingly independent algebras, Lemma 1.1 is no longer at our disposal. We have to replace (1.4) by the Burgholder-Gundy inequality from [PX1, JX1] (see also [JX2]) for constants):

$$\|\sum_{k=1}^n x_k\|_p \;\le\; cp\, \max\{\|\sum_{k=1}^n x_k^* x_k\|_{\frac{p}{2}}^{\frac{1}{2}}, \|\sum_{k=1}^n x_k x_k^*\|_{\frac{p}{2}}^{\frac{1}{2}}\} \,.$$

Here the underlying martingale structure is given by the von Neumann algebras B_{k-1} generated by $A_1, ..., A_{k-1}$. Let us denote by E_{k-1} the conditional on B_{k-1}. By assumption (and approximation) we have

$$E_{k-1}(x_k) \;=\; E_M(x_k)$$

Assuming $E_M(x_k) = 0$ for all k we deduce that $\sum_k x_k$ is a martingale and the Burgkholder-Gundy inequalities apply. With the iteration procedure from above this yields a constant Cp^2 for the upper estimate. ■

Remark 1.3. In the commutative case the best constant in the Burkholder Rosenthal inequality is of the order $p/1 + \ln p$. In view of this result the constant of order p seems reasonable. At the time of this writing we don't know whether this order of the constant is optimal. It seems that the best order of growth for Burkholder's inequality in the noncommutative case is unknown. Our estimate works with Cp^2.

Remark 1.4. a) As stated our result holds for σ-finite von Neumann algebras. It can easily be extended to arbitrary von Neumann algebras provided $E : N \to M$ is a faithful conditional expectation and E_{A_k} are normal conditional expectations satisfying the commutation relation $E_{A_k}E = E_{A_k}E = E$. Indeed, let ψ be a strictly semifinite weight on M, i.e. for a weight of the form $\psi = \sum_i s_i\phi_i s_i$ such that the s_i's are mutually orthogonal, $\sum_i s_i = 1$ and the states ϕ_i are faithful on $s_i M s_i$. Such a weight can always be constructed by choosing a maximal orthogonal family of states. For a finite subset $J \subset I$, we define $s_J = \sum_{i\in I} s_i$ and obtain an increasing family of projections such that $\lim_j s_J = 1$. Then we may consider the state $\phi_J = 1/|J| \sum_{i\in J} \phi_i \circ E$ on $s_J N s_J$ and see that the assumptions of Theorem 1.2 are satisfied for $A_{k,J} = s_J A_k s_J$. Moreover, for $x \in L_p(N)$ we find $x = \lim_J s_J x = \lim_J x s_J = \lim_J s_J x s_J$. For $p < \infty$ this is a norm limit. Thus the result extends to $L_p(N)$ by density.

b) There is also a standard procedure to reduce problems for arbitrary von Neumann algebras to von Neumann algebras with separable dual (see [GGMS, Appendix]). Indeed, let ϕ be a faithful normal state. Let $B \subset N$ countable subset of N. Then, we may consider the *-invariant subalgebra $\tilde{B}$ generated by $\sigma_t^\phi(E_{A_k}(A))_{k\in\mathbb{N},t\in\mathbb{Q}}$. The von Neumann algebra $M_A \subset N$ generated by $\tilde{B}$ satisfies Takesaki's criterion that $\sigma_t^\phi(M_B) \subset M_B$ and hence there is a ϕ-invariant conditional expectation. Thus there exists a net $(M_B)_{B\subset N \text{countable}}$ of von Neumann subalgebras of N which admit a unique ϕ-invariant conditional expectations E_{M_B} satisfying $E_{A_k}(M_B) \subset M_B$. By uniqueness we have $E_{M_{B_2}}|_{M_{B_1}} = E_{M_{B_1}}$ whenever $B_1 \subset B_2$. Therefore $L_p(N)$ is the direct limit of the complemented subspaces $L_p(M_B)$ and the assumptions of Theorem **??** are still satisfied for $(M_B \cap A_k)_{k\in\mathbb{N}}$.

As an illustration we prove our main result on random matrices.

Corollary 1.5. *Let $2 \le p < \infty$ and (f_{ij}) a matrix of independent random matrices in $L_p(N)$ defined on a probability space (Ω,μ) such that $\mathbb{E}(f_{ij}) = \int f_{ij} d\mu = 0$. Then*

$$\begin{aligned}
\frac{1}{Cp} \| \sum_{ij} f_{ij} \otimes e_{ij} \|_{L_p(\Omega;L_p(N\bar{\otimes}B(\ell_2)))} \;\le\; & \max \Bigg\{ \Big(\sum_{ij} \|f_{ij}\|_{L_p(N)}^p \Big)^{\frac{1}{p}}, \\
& \Big(\sum_j \| \Big(\sum_i \mathbb{E}(f_{ij}^* f_{ij}) \Big)^{\frac12} \|_p^p \Big)^{\frac1p}, \Big(\sum_i \| \Big(\sum_j \mathbb{E}(f_{ij} f_{ij}^*) \Big)^{\frac12} \|_p^p \Big)^{\frac1p} \Bigg\} \\
\le 2 \, \| \sum_{ij} f_{ij} \otimes e_{ij} \|_{L_p(\Omega;L_p(N\bar{\otimes}B(\ell_2)))} \, .
\end{aligned}$$

If the f_{ij} are independent scalar-valued mean 0 random variables, then

$$\begin{aligned}
& \| \sum_{ij} f_{ij} e_{ij} \|_p \\
& \sim_{Cp} \max \left\{ \Big(\sum_{ij} \|f_{ij}\|_p^p \Big)^{\frac1p}, \Big(\sum_j \Big(\sum_i \|f_{ij}\|_2^2 \Big)^{\frac p2} \Big)^{\frac1p}, \Big(\sum_i \Big(\sum_j \|f_{ij}\|_2^2 \Big)^{\frac p2} \Big)^{\frac1p} \right\} .
\end{aligned}$$

Proof. According to Remark 1.4 we may assume N is σ-finite N. Let ϕ_N be a normal faithful state ϕ_N on N and let ψ be a faithful normal state on $B(\ell_2)$. We deduce that

$\phi(x) = \int(\phi_N \otimes \psi)(x(\omega))d\mu(\omega)$ is a faithful normal state on $L_\infty(\Omega,\mu;N\bar{\otimes}B(\ell_2))$. Since the (f_{ij})'s are independent, we may assume that $\Omega = \prod_{ij}\Omega_{ij}$ such that f_{ij} is Ω_{ij} measurable. We define the algebras $A_{ij} = L_\infty(\Omega_{ij},\mu;N\bar{\otimes}B(\ell_2))$. Obviously, these algebras are independent over $1\otimes N\bar{\otimes}B(\ell_2) \subset L_\infty(\Omega, N\bar{\otimes}B(\ell_2))$. We use the lexicographical order on $\mathbb{N}\times\mathbb{N}$. Since $f_{ij}\otimes e_{ij} \in L_p(A_{ij}) = L_p(\Omega_{ij};L_p(N\bar{\otimes}B(\ell_2)))$ the assumptions of Theorem 1.2 are satisfied. Let us calculate the little square functions

$$\|\sum_{ij}\mathbb{E}((f_{ij}\otimes e_{ij})^*f_{ij}\otimes e_{ij})\|_{\frac{p}{2}} = \|\sum_{ij}\mathbb{E}(f_{ij}^*f_{ij})\otimes e_{ji}e_{ij})\|_{\frac{p}{2}} = \left(\sum_j \|\sum_i \mathbb{E}(f_{ij}^*f_{ij})\|_{\frac{p}{2}}^{\frac{p}{2}}\right)^{\frac{2}{p}} .$$

The same calculation applies for the second square function. The second chain of inequalities corresponds to the particular case $N=\mathbb{C}$. ■

2. The non-faithful case for $2 \le p < \infty$

Non-faithful filtrations occur very naturally in operator algebras. The easiest example is the filtration (M_n) in $B(\ell_2)$ given by matrix algebras M_n of matrices (a_{ij}) such that $a_{ij} = a_{ji} = 0$ for $i > n$. The notion of non-faithful copies of von Neumann algebras is important in the context of iterated ultraproducts of von Neumann algebras.

Let us fix the relevant notation. We will assume that $M \subset N$ is a weakly closed *-invariant subalgebra with unit e and a conditional expectation $E : N \to M$ with support e. In addition we assume that ϕ is a normal state on N, faithful on eNe such that $\phi = \phi|_M \circ E = \phi$. Now, we consider a sequence (A_k) of weakly closed *-invariant subalgebras of N containing M. Let us denote by r_k the units of A_k. We will say that the algebras A_k are *independent over* M if

i) The projections $s_k = r_k - e$ are mutual orthogonal.
ii) There are conditional expectations $F_k : s_kNs_k \to s_kA_ks_k$ with support s_k.
iii) $E_M(xy) = E_M(x)E_M(y)$ holds for all $x_k \in A_k$ and y in the *-algebra generated by $A_1, ..., A_{k-1}, A_{k+1}, ..., A_n$.

Let us observe that the sequence of von Neumann algebras $(eA_ke)_k$ are independent over M in the sense of the previous section. In the following we will say that $(eA_ke)_k$ is *faithfully independent over* E. This allows us to distinguish the notation from the previous section from the on defined above. For our exposition we may assume that n is finite and that the weakly closed subalgebras $B_k = s_kA_ks_k$ are σ-finite (see 3.5). Let us fix normal faithful states $\psi_1, ..., \psi_k$ on B_k. Then the state

$$\Phi(x) \;=\; \frac{1}{n+1}(\phi(x)+\sum_{k=1}^n \psi_k(F_k(x))) \tag{2.1}$$

is normal and faithful on N. It is easily checked that the support of $\psi_k \circ F_k$ is s_k. We obtain the new projection $f = e + \sum_k s_k$ and Φ is a normal faithful state on fNf. Thanks to our assumptions on the conditional expectations F_k, we may and will assume in the following that $f = 1$ because all the algebras A_k are contained in fNf. Moreover, by construction, the modular group of Φ may be calculated componentwise

$$\sigma_t^{\Phi}(x) \;=\; e\sigma_t^{\phi}(exe)e + \sum_{k=1}^n s_k \sigma_t^{\psi_k \circ F_k}(s_k x s_k)s_k \,. \tag{2.2}$$

By orthogonality of the s_k, we deduce that the von Neumann algebras A_k are Φ-invariant. Once this is established, we obtain Φ-invariant conditional expectations $E_k : N \to A_k$ such that $E_k(e) = e$. We refer to the beginning of section 2 on how to extend these conditional expectations to $L_p(N)$. The new ingredient for the non-faithful version of Rosenthal's inequality is a separate treatment of the corners.

Lemma 2.1. *Let $2 \le p < \infty$ and $x_k \in L_p(A_k)$. Then*

$$\|\sum_k s_k x_k e\|_p \;\le\; (1+Cp)^{\frac12} \;\max\{\|(s_k x_k e)\|_{\ell_p(L_p)}, \|(s_k x_k)\|_{L_p(M,E_M,\ell_2^c)}\} \,.$$

Proof. Let $x = \sum_k s_k x_k e$. By the orthogonality of the s_k, we obtain

$$\|x\|_p^2 \;=\; \|\sum_k e x_k^* s_k x_k e\|_{\frac p2} \;\le\; \|\sum_k E_M(x_k^* s_k x_k)\|_{\frac p2} + \|\sum_k e x_k^* s_k x_k e - E_M(x_k^* s_k x_k)\|_{\frac p2} \,.$$

We observe that $y_k \;=\; e x_k^* s_k x_k e - E_M(x_k^* s_k x_k) \in eA_k e$ and satisfies $E_M(y_k) = 0$. As observed before, the sequence $(eA_k e)$ is faithfully independent over M. Now, we follow the proof of Theorem 1.2. If $2 \le p \le 4$, we deduce from Lemma 1.1 that

$$\|\sum_k y_k\|_{\frac p2} \le 2\,\mathbb{E}\,\|\sum_k \varepsilon_k y_k\| \;\le\; 2\left(\sum_{k=1}^n \|y_k\|_{\frac p2}^{\frac p2}\right)^{\frac 2p} \;\le\; 2\left(\sum_{k=1}^n \|s_k x_k e\|_p^p\right)^{\frac 2p} \,.$$

For $4 < p < \infty$, we deduce from the fact that the y_k's are selfadjoint, Theorem 1.2 applied to $p/2$ and Lemma [JX1, Lemma 5.2] that

$$\|\sum_k y_k\| \;\le\; \frac{Cp}{2}\max\left\{\left(\sum_k \|y_k\|_{\frac p2}^{\frac p2}\right)^{\frac 2p}, \|\big(\sum_k E_M(y_k^* y_k)\big)^{\frac12}\|_{\frac p2}\right\}$$

$$\le \frac{Cp}{2} \max\left\{ 2\left(\sum_k \|s_k x_k e\|_p^p\right)^{\frac{2}{p}}, \|\sum_k E_M(|s_k x_k e|^4)\|_{\frac{p}{4}}^{\frac{1}{2}} \right\}$$

$$\le \frac{Cp}{2} \max\left\{ 2\left(\sum_k \|s_k x_k e\|_p^p\right)^{\frac{2}{p}}, \left(\sum_k \|s_k x_k e\|_p^p\right)^{\frac{1}{p-2}} \|\sum_k E_M(|s_k x_k e|^2)\|_{\frac{p}{2}}^{\frac{2(p/4-1)}{p-2}} \right\}.$$

The assertion follows by homogeneity. ■

The non-faithful version of Rosenthal's inequality for $p \ge 2$ reads as follows.

Theorem 2.2. *Let $2 \le p < \infty$. Let (A_k) be independent over M with units (r_k). Let $s_k = r_k - e$ above. Let $x_k \in L_p(A_k)$ and $y_k = x_k - E_M(x_k)$. Then*

$$\frac{1}{Cp} \|\sum_k x_k\|_p \le \max\{\|\sum_k E_M(x_k)\|_p, \|(y_k)\|_{\ell_p(L_p)}, \|(y_k)\|_{L_p(N,E_M,\ell_2^c)}, \|(y_k)\|_{L_p(N,E_M,\ell_2^r)}\}$$
$$\le 7\|\sum_k x_k\|_p .$$

Proof. We recall that the algebras $eA_ke \subset eNe$ are independent over E_M (in the sense of section 2). We need some easy facts. First $s_k y_k = s_k(x_k - E_M(x_k)) = s_k x_k$ because $E_M(x_k) \in eNe$. Similarly $x_k s_k = y_k s_k$. Theorem (1.2) implies in combination with Lemma 2.1 that

$$\|\sum_k x_k\|_p$$
$$\le \|\sum_k E_M(x_k)\|_p + \|\sum_k s_k x_k s_k\|_p + \|\sum_k s_k x_k e\|_p + \|\sum_k e x_k s_k\|_p + \|\sum_k e y_k e\|_p$$
$$\le \|\sum_{k=1}^n E_M(x_k)\|_p + \left(\sum_k \|s_k x_k s_k\|_p^p\right)^{\frac{1}{p}}$$
$$+ Cp\left(\sum_k \|e x_k s_k\|_p^p\right)^{\frac{1}{p}} + Cp\left(\sum_k \|s_k x_k e\|_p^p\right)^{\frac{1}{p}} + Cp\left(\sum_k \|e y_k e\|_p^p\right)^{\frac{1}{p}}$$
$$+ Cp\|(\sum_k E_M(x_k^* s_k x_k))^{\frac{1}{2}}\|_p + Cp\|(\sum_k E_M(y_k^* e y_k))^{\frac{1}{2}}\|_p$$
$$+ Cp\|(\sum_k E_M(x_k s_k x_k^*))^{\frac{1}{2}}\|_p + Cp\|(\sum_k E_M(y_k e y_k^*))^{\frac{1}{2}}\|_p$$
$$\le \|\sum_{k=1}^n E_M(x_k)\|_p + 4Cp\left(\sum_k \|y_k\|_p^p\right)^{\frac{1}{p}}$$

$$+ 3Cp\|(\sum_k E_M(y_k^* y_k))^{\frac{1}{2}}\|_p + 3Cp\|(\sum_k E_M(y_k y_k^*))^{\frac{1}{2}}\|_p.$$

This proves the upper estimate. E_M being a contraction implies $\|\sum_k E_M(x_k)\|_p \le \|\sum_k x_k\|_p$. The estimate

$$\left(\sum_k \|y_k\|_p^p\right)^{\frac{1}{p}} \;\le\; 2\|\sum_k x_k\|_p$$

holds by interpolation. By orthogonality we have

$$\begin{aligned} \|\sum_k E_M(y_k^* y_k)\|_{\frac{p}{2}}^{\frac{1}{2}} \;&=\; \|\sum_{kj} E_M(y_k^* y_j)\|_{\frac{p}{2}} \;=\; \|E_M(y^* y)\|_{\frac{p}{2}}^{\frac{1}{2}} \;\le\; \|y\|_p \\ &\le \|\sum_k x_k\|_p + \|\sum_k E_M(x_k)\|_p \;\le\; 2\|\sum_k x_k\|_p \,. \end{aligned}$$

This completes the proof of the lower estimate. ■

Example 2.3. Non-faithful independence occurs natural in the context of conditional expectation with respect to corners. Let N be a von Neumann algebra, e a projection and $r_1, ..., r_n$ be a family of orthogonal projections such that $e \le r_j$ and

$$r_j r_k \;=\; r_k r_j \;=\; e\,.$$

We also assume $\vee r_j = 1$. Let us consider

$$M \;=\; eNe \quad , \quad A_k \;=\; r_k N r_k\,.$$

The conditional expectation is given by $E(x) = exe$. Here F_k is the identity. We define $\hat{r}_k = \vee_{j\neq k} r_j$ and observe that $r_k \hat{r}_k \;=\; e$. The algebra B_k is given by $\hat{r}_k N \hat{r}_k$. Thus for $x \in A_k$ and $y \in B_k$ we have

$$E(xy) \;=\; exye \;=\; exr_k e_k ye \;=\; exeye \;=\; E(xy)\,.$$

This situation occurs for example on a tensor product $N = B^{\otimes n}$ where $e = f_1 \otimes \cdots \otimes f_n$ and $r_k \;=\; f_1 \otimes \cdots f_{k-1} \otimes 1 \otimes f_{k+1} \otimes \cdots \otimes f_n$.

3. A variation of Rosenthal's inequality using maximal functions

We will now discuss a version of Rosenthal's inequality where the term in $\ell_p(L_p(N))$ is replaced by $L_p(N;\ell_\infty)$. This is in perfect analogy with Burkholder's original inequality in the commutative case. Our argument is based on interpolation and, unfortunately, has a singularity for $p \to 2$. We need some facts on vector-valued L_p spaces with values in ℓ_q^n.

Let us recall the definition of the spaces $L_p(N;\ell_\infty^n)$ and $L_p(N;\ell_1^n)$ defined for finite sequences $(x_k)_{1\le k\le n}$, $n\in\mathbb{N}$, by

$$\|x\|_{L_p(N;\ell_\infty^n)} \;=\; \inf_{x_k \,=\, ay_kb} \|a\|_{2p}\sup_k \|y_k\|\|b\|_{2p}$$

and

$$\|x\|_{L_p(N;\ell_1^n)} \;=\; \inf_{x_k \,=\, \sum_j a_{kj}^*b_{kj}} \|\sum_{kj} a_{kj}^*a_{kj}\|_p^{\frac{1}{p}}\|\sum_{kj} b_{kj}^*b_{kj}\|_p^{\frac{1}{p}} \;.$$

For more information (see [Jun1]) and [JRX2] where the connection to decomposable maps is explained). We may then define

$$L_p(N;\ell_q^n) \;=\; [L_p(N;\ell_\infty^n), L_p(N;\ell_1^n)]_{\frac{1}{q}} \;.$$

We will write $L_p(N;\ell_q^n)$ if we want to emphasize the underlying von Neumann algebra. Some preliminary facts about these spaces are needed in the following.

Lemma 3.1. *Let N be von Neumann algebra and $x\in L_\infty(N;\ell_p^n)$ and $a,b\in L_{2p}$. Then*

$$\|axb\|_{\ell_p^n(L_p(N))} \;\le\; \|a\|_{2p}\|x\|_{L_\infty(\ell_p^n)}\|b\|_{2p} \;.$$

Proof. By trilinear interpolation it suffices to check this for $p=\infty$ (obvious) and for $p=1$. Given

$$x_k \;=\; \sum_{kj} v_{jk}^*w_{jk}$$

and $a,b\in L_2(N)$, we deduce from Hölder's inequality that

$$\begin{aligned}
\sum_{k=1}^n \|ax_kb\|_{L_1(N)} &= \sum_{k=1}^n \|\sum_j av_{jk}^*w_{jk}b\|_{L_1(N)} \\
&\le \left(\sum_{k=1}^n \|\sum_j av_{jk}^*v_{jk}a^*\|_1\right)^{\frac{1}{2}}\left(\sum_{k=1}^n \|\sum_j b^*w_{jk}^*w_{jk}b\|_1\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^n tr(\sum_j v_{jk}^*v_{jk}a^*a)\right)^{\frac{1}{2}}\left(\sum_{k=1}^n tr(\sum_j w_{jk}^*w_{jk}bb^*)\right)^{\frac{1}{2}} \\
&\le \|\sum_{kj} v_{jk}^*v_{jk}\|^{\frac{1}{2}}\|a\|_2\|\sum_{kj} w_{jk}^*w_{jk}\|^{\frac{1}{2}}\|b\|_2 \;.
\end{aligned}$$

■

Before we prove a stability result with respect to conditional expectations, we have to recall some facts about the Stinespring dilation theorem (see also [Jun1][JRX1]). Following [Ru] there exists a normal representation $\pi: M\to N\bar{\otimes}B(\ell_2)$ such that

$$E(x) \;=\; (1\otimes e_{11})\pi(x)(1\otimes e_{11}) \;.$$

(In the case of non-separable predual, we have to use a larger index set). Therefore, we may define the map $u : M \to N\bar{\otimes}C$ by $u(x) \;=\; \pi(x)(1 \otimes e_{11})$ which satisfies

$$E(y^*x) \;=\; u(y)^*u(x)\;. \tag{3.1}$$

Following [Jun1] this map has a natural extension $u_p : L_p(N) \to L_p(M; \ell_2^c)$ given by

$$u_p(xD^{\frac{1}{p}}) \;=\; \sum_k (1 \otimes e_{kk})u(x)D^{\frac{1}{p}} \otimes e_{1k}\;.$$

We clearly have

$$u_p(yD^{\frac{1}{p}})^*u_p(xD^{\frac{1}{p}}) \;=\; D^{\frac{1}{p}}E(y^*x)D^{\frac{1}{p}} \;=\; E_{\frac{p}{2}}((yD^{\frac{1}{p}})^*xD^{\frac{1}{p}})\;.$$

This shows that for $2 \le p \le \infty$ the map u_p extends to an isometry $L_p^c(M, E)$ such that $u_p(y)^*u_p(x) \;=\; E(y^*x)$. As usual we drop the index p.

Lemma 3.2. *Let* $(y_k)_{k=1}^n \subset L_1(N)$. *Then*

$$\|(y_k)\|_{L_1(N;\ell_\infty^n)} \;=\; \|(y_k)\|_{L_\infty(N;\ell_1^n)^*}\;.$$

Proof. Let $(y_k) \subset L_1(N)$ and let us assume that the corresponding functional $l : L_\infty(N; \ell_1^n)$ given by

$$l((x_k)) \;=\; \sum_k tr(y_kx_k)$$

has norm ≤ 1. Then, we apply the Hahn-Banach argument in [Jun1, Proposition 3.6] and find states ϕ and ψ such that

$$|tr(y_kab)| \;\le\; \phi(a^*a)^{\frac{1}{2}}\psi(b^*b)^{\frac{1}{2}}\;.$$

holds for all $1 \le k \le n$. By approximation we deduce for all $a, b \in N^{**}$ that

$$|\phi_{y_k}(ab)| \;\le\; \phi(a^*a)^{\frac{1}{2}}\psi(b^*b)^{\frac{1}{2}}\;.$$

where ϕ_{y_k} is the unique extension of the linear functional $\phi_{y_k}(n) = tr(y_kn)$ to N^{**}. Let z be the central projection in N^{**} such that $N = zN^{**}$. Then $z.\phi_{y_k}.z = \phi_{y_k}$. This yields

$$|\phi_{y_k}(ab)| \;\le\; \phi(za^*az)^{\frac{1}{2}}\psi(zb^*bz)^{\frac{1}{2}}\;.$$

However, $z.\phi$ $(z.\psi)$ is the normal part of ϕ (ψ, respectively). This implies that there are d_1 and $d_2 \in L_1(N)$ such that

$$|tr(y_kab)| \;\le\; tr(d_1a^*a)^{\frac{1}{2}}tr(d_2b^*a)^{\frac{1}{2}}\;.$$

Now, the proof in [Jun1, Propositon 3.6] may be completed as stated and yields

$$\|(y_k)\|_{L_\infty(N;\ell_1^n)^*} \;\ge\; \|(y_k)\|_{L_1(N;\ell_\infty^n)}\;.$$

The converse inequality is elementary. ■

Lemma 3.3. *Let $M \subset N$, ϕ be a faithful normal state on N and $E : N \to M$ be a conditional expectation such that $\phi = \phi|_M \circ E$. Then*

$$L_p(N; \ell_q^n) \subset L_p(M; \ell_q^n)$$

holds isometrically and E extends to a contraction on $L_p(M; \ell_q^n)$.

Proof. We first observe that by definition the inclusion maps

$$L_p(M; \ell_\infty^n) \subset L_p(N; \ell_\infty^n) \quad \text{and} \quad L_p(M; \ell_1^n) \subset L_p(N; \ell_1^n)$$

are contractive. Hence by interpolation the inclusion

$$L_p(M; \ell_q^n) \subset L_p(N; \ell_q^n)$$

is contractive. The inclusion will certainly be isometric if we can show $E \otimes id : L_p(N; \ell_q^n) \to L_p(M; \ell_q^n)$ is contractive. By interpolation it suffices to shows this for $q = \infty$ and $q = 1$. We start with the latter case and consider

$$x_k \;=\; \sum_j a_{kj}^* b_{kj} \,.$$

Then we deduce from (3.1) that

$$E(x_k) \;=\; \sum_j E(a_{kj}^* b_{kj}) \;=\; \sum_j u(a_{kj})^* u(b_{kj}) \,.$$

Since E is a contraction $L_p(N)$ we deduce that

$$\begin{aligned} \|(E(x_k))\|_{L_p(M;\ell_1^n)} &\le \|\sum_{kj} u(a_{kj})^* u(a_{kj})\|_p^{\frac{1}{2}} \|\sum_{kj} u(b_{kj})^* u(b_{kj})\|_p^{\frac{1}{2}} \\ &= \|\sum_{kj} E(a_{kj}^* a_{kj})\|_p^{\frac{1}{2}} \|\sum_{kj} E(b_{kj}^* b_{kj})\|_p^{\frac{1}{2}} \\ &\le \|\sum_{kj} a_{kj}^* a_{kj}\|_p^{\frac{1}{2}} \|\sum_{kj} b_{kj}^* b_{kj}\|_p^{\frac{1}{2}} \,. \end{aligned}$$

Taking the infimum over all decompositions for (x_k) yields the assertion. By duality (see [Jun1, Proposition 3.6]) we deduce the assertion for $L_{p'}(N; \ell_\infty^n)$ in the range $1 < p' \le \infty$. For $p' = \infty$ we apply Lemma 3.2. Then $E^* \otimes id$ is contraction on $L_\infty(N; \ell_1^n)^*$. The restriction to $L_1(N; \ell_\infty^n)$ is exactly $E \otimes id$. ■

Proposition 3.4. *Let $1 \le p \le \infty$. Then*

$$L_p(N; \ell_p^n) \;=\; \ell_p^n(L_p(N))$$

holds isometrically.

Proof. We will first prove that the inclusion map

$$L_p(N;\ell_p^n) \subset \ell_p^n(L_p(N)) \tag{3.2}$$

is contractive for finite von Neumann algebras N. Let $x_k = f_k(1/p)$ where $(f_k)_{k=1}^n$ is a family of analytic functions in $L_p(N)$ such that

$$\sup_t \|(f_k(it)\|_{L_p(N;\ell_\infty^n)} \le 1-\varepsilon \quad \text{and} \quad \sup_t \|(f_k(1+it)\|_{L_p(N;\ell_1^n)} \le 1-\varepsilon \ .$$

By the continuous selection theorem, we may find continuous functions a, b and y defined on $i\mathbb{R}$ satisfying

$$f_k(it) \ = \ a(it)y_k(it)b(it)$$

such that $\|a(it)\|_{2p} \le 1$, $\sup_k \|y_k(it)\| \le 1$ and $\|b(it)\|_{2p} \le 1$. We note that for $z = 1+it$ we may write

$$f_k(z) \ = \ \sum_j a_{jk}^* b_{jk}$$

such that

$$\|\sum_{kj} a_{jk}^* a_{jk}\|_p \le 1 \quad \text{and} \quad \|\sum_{kj} b_{jk}^* b_{jk}\|_p \le 1 \ .$$

Using $a = (\sum\limits_{kj} a_{kj}^* a_{kj})^{\frac{1}{2}}$ and $v_{kj} \ = \ a^{-\frac{1}{2}} a_{kj}$, we see that

$$f_k(z) \ = \ a g_k b \ = \ a(\sum_{kj} v_{kj}^* w_{kj}) b$$

where $(g_k) \in L_\infty(N;\ell_1^n)$. Applying the continuous selection theorem again we find continuous maps a, b and y on $1+i\mathbb{R}$ such that

$$f_k(z) \ = \ a(z)y_k(z)b(z) \ .$$

We apply [PX2] and obtain an analytic invertible function $\alpha, \beta : \Omega \to L_{2p}(N)$ such that

$$\alpha(z)\alpha(z)^* \ = \ a(z)a(z)^* + \delta 1 \quad \text{and} \quad \beta(z)^*\beta(z) \ = \ b(z)^*b(z) + \delta 1$$

for all $z \in \partial\Omega$. Then we may define

$$y_k(z) \ = \ \alpha(z)^{-1} f_k(z) \beta(z)^{-1} \ .$$

For $z \in \partial\Omega$ we deduce from $a(z)^*a(z) \le \alpha(z)^*\alpha(z)$ and $b(z)^*b(z) \le \beta(z)^*\beta(z)$ that there are contractions $v(z)$ and $w(z)$ satisfying $a(z) \ = \ \alpha(z)v(z)$, $b(z) \ = \ w(z)\beta(z)$. This implies that

$$\|(y_k(1/p))\|_{L_\infty(N;\ell_p^n)} \ \le \ 1 \ .$$

By the definition of α and β is also clear that

$$\|\alpha(1/p)\|_p \ \le \ (1+\delta)^{\frac{1}{2}} \quad \text{and} \quad \|\beta(1/p)\|_p \ \le \ (1+\delta)^{\frac{1}{2}} \ .$$

According to Lemma 3.1 we deduce

$$\|(x_k)\|_{\ell_p^n(L_p(N))} \;\le\; \|\alpha(1/p)\|_{2p}\|(y_k(1/p))\|_{L_\infty(\ell_p^n)}\|\beta(1/p)\|_{2p} \;\le\; (1+\delta)\,.$$

Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary, the assertion is proved in the finite case. For the general σ-finite case we use Haagerup's crossed product construction and consider $\mathcal{N} = N \rtimes_{\sigma_t^\phi} G$ where $G = \bigcup_n 2^{-n}\mathbb{Z}$ is the group of duadic rationals. There is a conditional expectation $F : \mathcal{N} \to N$ given by the identity element. Then $\hat{\phi} \;=\; \phi\circ F$ is a normal faithful state. We refer for [JX3] for the fact that there is an increasing sequence $\mathcal{N}_m$ of $\hat{\phi}$ invariant finite von Neumann algebras such that the corresponding conditional expectations $\mathcal{E}_m : \mathcal{N} \to \mathcal{N}_m$ converge to the identity in the strong operator topology. Then $\mathcal{E}_m(x)$ converges to x in norm for every element $x \in L_p(\mathcal{N})$. According to Lemma 3.3 and our previous argument, we deduce that

$$\left(\sum_{k=1}^n \|F\mathcal{E}_m(x_k)\|_{L_p(\mathcal{N})}^p\right)^{\frac{1}{p}} \;\le\; \|(x_k)\|_{L_p(\ell_p^n)}\,. \tag{3.3}$$

Thus passing to the limit shows that the inclusion (3.2) is contractive. Applying the same argument for p' we also have

$$L_{p'}(N;\ell_{p'}^n) \subset \ell_{p'}^n(L_{p'}(N))\,.$$

By duality

$$\ell_p^n(L_p(N)) \subset L_{p'}(N;\ell_{p'}^n)^*\,.$$

By interpolation and [Jun1, Proposition 3.6] we have for $1 < p \le \infty$ that

$$\begin{aligned} L_{p'}(N;\ell_{p'}^n)^* \;&=\; [L_{p'}(N;\ell_\infty^n), L_{p'}(N;\ell_1^n)]_{\frac{1}{p'}}^* \;=\; [L_{p'}(N;\ell_\infty^n)^*, L_{p'}(N;\ell_1^n)^*]_{\frac{1}{p'}} \\ &= [L_p(N;\ell_1^n), L_p(N;\ell_\infty^n)]_{1-\frac{1}{p}} \;=\; [L_p(N;\ell_\infty^n), L_p(N;\ell_1^n)]_{\frac{1}{p}} \;=\; L_p(N;\ell_p^n)\,. \end{aligned}$$

Here we have used the equality $[L_{p'}(N;\ell_\infty^n)^*, L_{p'}(N;\ell_1^n)^*]^{\frac{1}{p'}} \;=\; [L_{p'}(N;\ell_\infty^n)^*, L_{p'}(N;\ell_1^n)^*]_{\frac{1}{p'}}$ which follows immediately from the fact that topological all these space coincide with $(L_{p'}(N)^*)^n$ and Bergh's theorem [Be]. For $p = 1$ the inclusion $\ell_1^n(L_1) \subset L_1(\ell_1^n)$ follows from the fact that the space $L_1(\ell_1^n)$ is normed (see [Pis1] for a similar argument). ■

Remark 3.5. *Motivated from Pisier's theory of vector valued L_p spaces we introduce*

$$|||\,(x_k)\,|||_{pq} \;=\; \inf_{x_k = a y_k b} \|a\|_{2p}\|(y_k)\|_{L_\infty(N;\ell_q^n)}\|b\|_{2p} \;=\; \|(x_k)\|_{L_p(N;\ell_q^n)}\,.$$

Then we have

$$|||\,(x_k)\,|||_{pq} \;=\; \|(x_k)\|_{L_p(\ell_q^n)}\,.$$

Proof. We refer to [Pis1] for the argument showing showing that $|||\,(x_k)\,|||_{pq}$ is a norm. The proof above shows that for a finite von Neumann algebra N, we have

$$|||\,(x_k)\,|||_{pq} \;\le\; \|(x_k)\|_{L_p(N,\ell_q^n)}\,.$$

In the general case, we deduce from (3.3) that for every $\gamma > 0$, we may find $x_{k,\gamma} = F\mathcal{E}_m(x_k)$ such that $\|x_{k,\gamma} - x_k\| \;\le\; \gamma$ and

$$|||\,(x_{k,\delta})\,|||_{pq} \;\le\; (1+\varepsilon)\|(x_k)\|_{L_p(N;\ell_q^n)}\,.$$

Let $\delta > 0$. Define $z_{k,0} = x_{k,\delta}$ and $z_{k,j} = x_{k,\delta 2^{-(j+1)}} - x_{k,\delta 2^{-j}}$ for $j \ge 1$. Then we have $x_k = x_{k,\delta} + x_k - x_{k,\delta} = \sum_{j\ge 0} z_{k,j}$. This implies with the triangle inequality

$$\begin{aligned}
|||\,(x_k)\,|||_{pq} &= |||\,(\sum_{j\ge 0} z_{k,j})\,|||_{pq} \;=\; |||\,(x_{k,\delta}) + \sum_{j\ge 1}(z_{k,j})\,|||_{pq} \;\le\; |||\,(x_{k,\delta})\,|||_{p,q} + |||\,\sum_{j\ge 1}(z_{k,j})\,|||_{pq} \\
&\le |||\,(x_{k,\delta})\,|||_{p,q} + \sum_{k=1}^{n} \|\sum_{j} z_{k,j}\|_{L_p} \;\le\; (1+\varepsilon)\|(x_k)\|_{L_p(\ell_q^n)} + \sum_{j\ge 0} n\delta(2^{-(j+1)} + 2^{-(j)}) \\
&\le (1+\varepsilon)\|(x_k)\|_{L_p(\ell_q^n)} + 3n\delta\,.
\end{aligned}$$

Since $\delta > 0$ and $\varepsilon > 0$ we deduce

$$|||\,(x_k)\,|||_{pq} \;\le\; \|(x_k)\|_{L_p(\ell_q^n)}\,.$$

The converse inequality follows from the contractive inclusion

$$L_{2p}(N)L_\infty(N;\ell_q^n)L_{2p}(N) \subset L_p(N;\ell_q^n)\,.$$

Indeed, this is obvious for $q = 1$ and $q = \infty$ and hence valid for all q by (trilinear) interpolation. ■

Lemma 3.6. *Let $1 \le q \le p, r \le \infty$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and $(x_k) \subset L_p(N)$. Then*

$$\|(x_k)\|_{L_p(N;\ell_q^n)} \;=\; \sup_{\|a\|_{2r}\|b\|_{2r}\le 1} \left(\sum_{k=1}^{n} \|ax_kb\|_q^q\right)^{\frac{1}{q}}\,.$$

Proof. Given an element (x_k) we apply Remark 3.5 and write

$$x_k \;=\; a_1 y_k b_1$$

such that

$$\|a_1\|_{2p}\|(y_k)\|_{L_\infty(N;\ell_q^n)}\|b_1\|_{2p} \;\le\; (1+\varepsilon)\|(x_k)\|_{L_p(N;\ell_q^n)}\,.$$

Thus for $a \in L_{2r}$ and $b \in L_{2r}$, we deduce from Lemma 3.1 that

$$\begin{aligned}
\|(ax_kb)\|_{\ell_q^n(L_q(N))} \;&=\; \|(aa_1y_kb_1b)\|_{\ell_q^n(L_q(N))} \;\le\; \|aa_1\|_{2q}\|y_k\|_{L_\infty(N;\ell_q^n)}\|b_1b\|_{2q} \\
&\le \|a\|_{2r}\|a_1\|_{2p}\|y_k\|_{L_\infty(N;\ell_q^n)}\|b_1\|_{2p}\|b\|_{2r} \;\le\; (1+\varepsilon)\|(x_k)\|_{L_p(\ell_q^n)}\|a\|_{2r}\|b\|_{2r}\,.
\end{aligned}$$

In order to prove the converse inequality we consider an element $(y_k) \in L_{p'}(N;\ell_{q'}^n)$ of norm less than one. According to Remark 3.5 we may write

$$y_k \;=\; az_kb$$

such that $\|a\|_{2p'} \le 1$, $\|(z_k)\|_{L_\infty(N;\ell_{q'}^n)} \le 1$ and $\|b\|_{2p'} \le 1$. We may decompose $a = a_1a_2$, $b = b_2b_1$ such that

$$\max\{\|a_1\|_{2q'}, \|a_2\|_{2r}\} \le 1 \quad \text{and} \quad \max\{\|b_2\|_{2q'}\|b_1\|_{2r}\} \le 1 \,.$$

According to Proposition 3.4, we deduce that

$$\|(a_1z_kb_1)\|_{\ell_{q'}^n(L_{q'}(N))} \le 1 \,.$$

This implies with Hölder's inequality that

$$\begin{aligned} |\sum_{k=1}^n tr(y_k^*x_k)| \;&=\; |\sum_{k=1}^n tr((a_2z_kb_2)^*x_k)| \;=\; |\sum_{k=1}^n tr((z_k^*a_2^*x_kb_2)| \\ &\le \left(\sum_{k=1}^n \|a_2^*x_kb_2^*\|_q^q\right)^{\frac{1}{q}} \|(z_k)\|_{\ell_{q'}^n(L_{q'}(N))} \;\le\; \left(\sum_{k=1}^n \|a_2^*x_kb_2^*\|_q^q\right)^{\frac{1}{q}} \,. \end{aligned}$$

Therefore, we have

$$\|(x_k)\|_{L_p(\ell_q^n)} = \sup_{\|(y_k)\|_{L_{p'}(\ell_{q'}^n)}\le 1} |\sum_{k=1}^n tr(y_k^*x_k)| \;\le\; \sup_{\|a_2\|_{2r}\|b_2\|_{2r}\le 1} \left(\sum_{k=1}^n \|a_2x_kb_2\|_q^q\right)^{\frac{1}{q}}$$

and the assertion is proved. ■

Corollary 3.7. *Let $2 \le p \le \infty$. Then*

$$[L_p(N,(\ell_2^n)^c), L_p(N,(\ell_2^n)^r)]_{\frac{1}{2}} \subset L_p(N;\ell_2^n) \,.$$

Proof. Let $\frac{1}{2} = \frac{1}{p} + \frac{1}{r}$. We use trilinear interpolation and consider the map

$$T(a,(x_k),b) \;=\; (ax_kb) \,.$$

First, we note that

$$T : L_\infty(N) \times L_p(N,(\ell_2^n)^c) \times L_r(N) \to \ell_2^n(L_2(N))$$

is a contraction because

$$\begin{aligned} \sum_{k=1}^n \|ax_kb\|_2^2 \le \sum_{k=1}^n tr(b^*x_k^*x_kb) \;&=\; tr((\sum_{k=1}^n x_k^*x_k)bb^*) \\ &\le \|\sum_{k=1}^n x_k^*x_k\|_{\frac{p}{2}} \|bb^*\|_r \;=\; \|(x_k)\|_{L_p(N;\ell_2^c)}^2 \|b\|_{2r}^2 \,. \end{aligned}$$

Similarly, we see that

$$T : L_r(N) \times L_p(N, (\ell_2^n)^r) \times L_\infty(N) \to \ell_2^n(L_2(N))$$

is a contraction. Thus by interpolation

$$T : L_{2r}(N) \times [L_p(N, (\ell_2^n)^c), L_p(N, (\ell_2^n)^r)]_{\frac{1}{2}} \times L_{2r}(N) \to \ell_2^n(L_2(N)) \,.$$

Lemma 3.6 implies the assertion. ∎

Theorem 3.8. *Let $2 < p < \infty$, $n \in \mathbb{N} \cup \{\infty\}$ and $(A_k)_{1 \le k \le n}$ be independent over M. Let $x_k \in L_p(A_k)$ with $E_M(x_k) = 0$. Then*

$$\|\sum_{k=1}^n x_k\|_p \;\le\; \max\{(2Cp)^{\frac{p}{p-2}}\|(x_k)\|_{L_p(N;\ell_\infty)}, Cp\|(x_k)\|_{L_p(N,E;\ell_2^c)}, Cp\|(x_k)\|_{L_p(N,E;\ell_2^r)}\} \,.$$

Here C is the constant from Theorem (1.2)*. For $p \ge 4$*

$$\|\sum_{k=1}^n x_k\|_p \;\le\; \tilde{C}p \max\{\|(x_k)\|_{L_p(\ell_\infty)}, \|(x_k)\|_{L_p(N,E;\ell_2^c)}, \|(x_k)\|_{L_p(N,E;\ell_2^r)}\}$$

holds for an absolute constant $\tilde{C}$.

Proof. Let us denote by C_1 the constant from Theorem (1.2). Let us assume that

$$\left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac{1}{p}} < \max\{\|(x_k)\|_{L_p(N,E;\ell_2^c)}, \|(x_k)\|_{L_p(N,E;\ell_2^r)}\} \,.$$

Then Theorem (1.2) implies

$$\|\sum_k x_k\|_p \;\le\; C_1 p \;\max\{\|(x_k)\|_{L_p(N,E;\ell_2^c)}, \|(x_k)\|_{L_p(N,E;\ell_2^r)}\} \,.$$

Let us consider the second case

$$\max\{\|(x_k)\|_{L_p(N,E;\ell_2^c)}, \|(x_k)\|_{L_p(N,E;\ell_2^r)}\} \;\le\; \left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac{1}{p}} \,.$$

Then Theorem 1.2 implies

$$\|\sum_{k=1}^n x_k\|_p \;\le\; C_1 p \left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac{1}{p}} \,.$$

Let θ such that $\frac{1}{p} \;=\; \frac{1-\theta}{\infty} + \frac{\theta}{2}$. From the reiteration theorem [BL], we deduce that

$$L_p(N;\ell_p^n) \;=\; [L_p(N;\ell_\infty^n), L_p(N;\ell_2^n)]_\theta \,.$$

This implies

$$\left(\sum_{k=1}^{n}\|x_k\|_p^p\right)^{\frac{1}{p}} \le \|(x_k)\|_{L_p(N;\ell_\infty^n)}^{1-\theta}\|(x_k)\|_{L_p(N;\ell_2^n)}^{\theta} .$$

We recall from the proof of Theorem 1.2 that

$$\max\{\|(\sum_{k=1}^{n}x_k^*x_k)^{\frac{1}{2}}\|_p, \|(\sum_{k=1}^{n}x_kx_k^*)^{\frac{1}{2}}\|_p\} \;\le\; 2\|\sum_{k=1}^{n}x_k\|_p .$$

According to Corollary 3.7 this implies

$$\|(x_k)\|_{L_p(N;\ell_2^n)} \;\le\; 2\|\sum_{k=1}^{n}x_k\|_p .$$

Combining these estimates we find (after cancellation) that

$$\|\sum_{k=1}^{n}x_k\|_p \;\le\; (2^\theta C_1 p)^{\frac{1}{1-\theta}}\|(x_k)\|_{L_p(\ell_\infty^n)} . \tag{3.4}$$

Since $\theta=\frac{2}{p}\le 1$ we deduce the first assertion. If moreover $p\ge 4$, we have

$$p^{\frac{p}{p-2}} \;=\; p^{1+\frac{2}{p-2}} \;\le\; p^{1+\frac{4}{p}} \;\le\; e^{\frac{4}{e}}p$$

and the assertion is follows. ∎

Remark 3.9. *The same improvement is also true in the context of Burkholder's inequality for martingales:*

$$\|\sum_k d_k\|_p \le \max\{(Cp^2)^{\frac{p}{p-2}}\|(d_k)\|_{L_p(N;\ell_\infty)}, Cp^2\|x\|_{h_p^c}, Cp^2\|x\|_{h_p^r}\}$$

where $\|x\|_{h_p^c}=\|(\sum_k E_{k-1}(d_k^*d_k))^{1/2}\|_p$ *and* $\|x\|_{h_p^r}=\|x^*\|_{h_p^c}$ *are the 'little square functions'.*

Proof. The only difference here is that we have to use the lower estimate in the square function inequality (see [JX2] for constant:)

$$\|(d_k)\|_{L_p(\ell_2)} \le \|(d_k)\|_{L_p(N;\ell_2^c)}^{\frac{1}{2}}\|(d_k)\|_{L_p(N;\ell_2^r)}^{\frac{1}{2}} \;\le\; cp\|\sum_k d_k\|_p .$$

We may assume again that

$$\max\{\|\big(\sum_k E_{k-1}(d_k^*d_k)\big)^{\frac{1}{2}}\|_p, \|\big(\sum_k E_{k-1}(d_kd_k^*)\big)^{\frac{1}{2}}\|_p\} \;\le\; \left(\sum_k\|d_k\|_p^p\right)^{\frac{1}{p}} .$$

In this situation (3.4) has to be replaced by

$$\|\sum_{k=1}^n d_k\|_p \le ((cp)^\theta C p^2)^{\frac{1}{1-\theta}} \|(x_k)\|_{L_p(\ell_\infty^n)} \;\le\; (Cce^{2/e}p^2)^{\frac{p}{p-2}} \|(x_k)\|_{L_p(\ell_\infty^n)} \,. \qquad \blacksquare$$

4. Rosenthal's inequality for $1 < p \le 2$

We will now investigate the dual version of Rosenthal's inequality for $1 < p \le 2$. In the following we will assume as before that $M \subset N$ is a subalgebra, invariant under modular group σ_t^ϕ of a normal faithful state ϕ.

Lemma 4.1. *Let $2 \le p < \infty$ and $2 \le r \le \infty$ such that $\frac{1}{2} = \frac{1}{p} + \frac{1}{r}$. The dual space of $L_p(N, E_M, \ell_2^c)$ consists of all sequences $(x_k) \subset L_{p'}(M)$ such that there is an $a \in L_r(M)$ and $(y_k) \subset \ell_2(L_2(M))$ and*

$$x_k \;=\; y_k a \,.$$

Moreover,

$$\|(x_k)\|_{\overline{L_p(N,E_M,\ell_2^c)}^*} \;=\; \inf_{x_k = y_k a} \left(\sum_{k=1}^\infty \|y_k\|_2^2 \right)^{\frac{1}{2}} \|a\|_r \,.$$

Let D be the density of ϕ. The space of finite sequences $(z_k D^{\frac{1}{p'}})$ is dense in $\overline{L_p(N, E_M, \ell_2^c)}^*$ and for those elements

$$\|(z_k D^{\frac{1}{p'}})\|_{\overline{L_p(N,E_M,\ell_2^c)}^*} \;=\; \|\sum_k D^{\frac{1}{p'}} E_M(z_k^* z_k) D^{\frac{1}{p'}}\|_{\frac{p'}{2}}^{\frac{1}{2}} \,.$$

Proof. According to Remark (1.4) we may assume that M is σ-finite and even that N has separable dual. Let us recall the map $u_p : MD^{\frac{1}{p}} \to L_p(M, \ell_2^c)$ constructed in (3.1) which satisfies

$$u_p(xD^{\frac{1}{p}})^* u(yD^{\frac{1}{p}}) \;=\; D^{\frac{1}{p}} E_M(x^* y) D^{\frac{1}{p}} \,.$$

This means that $u : L_p(N, E_M) \to L_p(N, \ell_2^c)$ is an isometric isomorphism when restricted to $MD^{\frac{1}{p}}$. For $2 \le p < \infty$ this space is dense in $L_p(M)$. We consider $M \bar{\otimes} C$ as a M-valued Hilbert C^*-module with the M-valued scalar product $\langle (x_k), (y_k) \rangle = \sum_k x_k^* y_k$. The map constructed in [Jun1] using Kasparov's dilation technique (see [Lan]) has the additional advantage that there exists a right M-module map $P : M \bar{\otimes} C \to M \bar{\otimes} C$ with $P^2 = P$ such that $u(N)$ is strongly dense in $P(M \bar{\otimes} C)$ and

$$\langle P((x_k)), (x_k) \rangle \;=\; \langle (x_k), P(x_k) \rangle \;=\; \langle P(x_k), P(x_k) \rangle \;\le\; \langle (x_k), (x_k) \rangle \tag{4.1}$$

Indeed, P is a the projection onto a submodule (see [Lan] for more details, see also [JS] for a general treatment without assuming N_* separable based on [Pas]). Thus for $2 \le p < \infty$ the map u_p extends to an isometric isomorphism and P extends to a contractive projection from $L_p(M, \ell_2^c)$ to the image u_p. In [Jun1] we defined $L_p(N, E_M)$ as the closure of $ND^{\frac{1}{p}}$ with respect to the norm $\|D^{\frac{1}{p}}E(x^*x)D^{\frac{1}{p}}\|_{p/2}^{1/2}$ for all $1 \le p \le \infty$. We may also consider $u_p \otimes id : L_p(M, E_M; \ell_2^c) \to L_p(M, \ell_2^c(\mathbb{N}^2))$ and see that

$$(u_p \otimes id)((x_j))^*(u_p \otimes id)((x_j)) \;=\; \sum_j u_p(x_j)^* u_p(x_j) \;=\; \sum_j E_p(x_j^* x_j)\,.$$

As above $P \otimes id : L_p(M, \ell_2^c(\mathbb{N}^2)) \to L_p(M, \ell_2(\mathbb{N}^2))$ extends to a projection, because we have

$$(P \otimes id)((x_{jk}))^*(P \otimes id)((x_{jk})) \;=\; \sum_j P(x_{j,k})^* P(x_{j,k}) \;\le\; \sum_{j,k} x_{jk}^* x_{jk}\,.$$

We may hence define $L_p(M, E_M; \ell_2^c)$ as the closure of the space of finite sequence $(w_j D^{\frac{1}{p}})$ with respect to the norm $\|\sum_j D^{\frac{1}{p}} E(w_j^* w_j) D^{\frac{1}{p}}\|_{p/2}^{1/2}$ for all $1 \le p < \infty$. By density this is consistent with the the definition given above for $p \ge 2$. Then $u_p \otimes id$ defines a isometric isomorphism onto a complemented subspace of $L_p(M, \ell_2(\mathbb{N}^2))$ for all values $1 \le p \le \infty$. Let $1 \le p < \infty$. Then we have $L_p(M, \ell_2^c(\mathbb{N}^2))^* = L_{p'}(M, \ell_2^c(\mathbb{N}^2)$. Therefore every functional $\psi : L_p(M, E_M; \ell_2^c) \to \mathbb{C}$ is given by

$$\psi_y((x_j)) \;=\; \sum_{jk} tr(y_{jk}^* u_p(x_j)_k)$$

and $\|\sum_{jk} y_{jk}^* y_{jk}\|_{p'/2}^{1/2} = \|\psi\|$. However, finite sequences of the form $y_{jk} = z_{jk} D^{\frac{1}{p'}}$ are dense in $L_{p'}(M, \ell_2^c(\mathbb{N}^2))$. For such an element we find (using (4.1))

$$\begin{aligned}
\psi_y((v_j D^{\frac{1}{p}})) &= \sum_j \sum_k tr(D^{\frac{1}{p'}} z_{jk}^* u(x_j)_k D^{\frac{1}{p}}) \;=\; \sum_j \sum_k \phi(z_{jk}^* P(u(v_j))_k) \\
&= \sum_j \phi(\langle (z_{jk})_k, Pu(v_j)\rangle) \;=\; \sum_j \phi(\langle P(z_{jk}), u(v_j)\rangle)\,.
\end{aligned}$$

This means that $P((z_{jk})_k)$ is in the range of u and we find an element $w_j \in M$ such that

$$\begin{aligned}
\psi_y((v_j D^{\frac{1}{p}})) &= \sum_j \phi(\langle P((z_{j,k})_k), u(v_j)\rangle) \;=\; \sum_j \phi(u(w_j)^* u(y_j)) \\
&= \sum_j \phi(E(w_j^* y_j)) \;=\; \sum_j tr(D^{\frac{1}{p'}} w_j^* y_j D^{\frac{1}{p}})\,.
\end{aligned}$$

Recall that only finitely many w_j's are non-zero and hence ψ_y has a unique extension as a linear functional (see also [Jun1, Proposition 2.15]). This shows that $\overline{L_p(N, E_M, \ell_2^c)}^*$ is the norm closure of finite sequences $(w_jD^{1/p'})$ and the norm of such a sequence satisfies

$$\|(w_jD^{\frac{1}{p'}})\|^2_{L_{p'}(N,E_M,\ell_2^c)} \;=\; \|\sum_j D^{\frac{1}{p'}}E(w_j^*w_j)D^{\frac{1}{p'}}\|_{p'/2} \;=\; \|\sum_j D^{\frac{1}{p'}}\langle P(z_j),P(z_j)\rangle D^{\frac{1}{p'}}\|_{p'/2}$$
$$\le \|\sum_j D^{\frac{1}{p'}}\sum_k z_{j,k}^*z_{j,k}D^{\frac{1}{p'}}\|_{p'/2} \;=\; \|(y_{j,k})\|^2_{L_{p'}(M,\ell_2(\mathbb{N}^2))}\,.$$

Thus we have identified the dual of $L_p(N,E_M,\ell_2^c)$ as the closure of finite sequences $(w_jD^{\frac{1}{p'}})$ in $L_{p'}(N,E_M,\ell_2^c)$ with respect to the antilinear duality given by the trace. Let $1\le p'\le 2\le p$ and $(w_j)\subset N$ be a finite sequence. We define

$$a \;=\; \sum_{j=1}^n D^{\frac{1}{p'}}E_M(w_j^*w_j)D^{\frac{1}{p'}}\,.$$

We may assume that $tr(a^{p'/2})=1$. Let $\delta>0$ and define $d=a+\delta D^{2/p'}$. Note that $\|d\|^{p'/2}_{p'/2}\le 1+\delta$. Since $D^{2/p'}\le\delta^{-1}d$, we find $v\in M$ such that $D^{1/p'}=vd^{1/2}$. Then, we define

$$d_j \;=\; w_jD^{\frac{1}{p'}}d^{-\frac{p'}{2r}} \;=\; w_jvd^{\frac12-\frac{p'}{2r}} \;=\; w_jvd^{\frac{p'}{4}}\,.$$

Note that $w_jD^{1/p'}=d_jd^{p'/2r}$ and $\|d^{p'/2r}\|_r^r=\|d^{p'/2}\|^{p'/2}_{p'/2}\le(1+\delta)$. On the other hand we have

$$d^{\frac12}v^*\sum_j E_M(w_j^*w_j)vd^{\frac12} \;=\; D^{\frac{1}{p'}}\sum_j E_M(w_j^*w_j)D^{\frac{1}{p'}} \;\le\; d\,.$$

Since d has full support we deduce $\sum_j v^*E_M(w_j^*w_j)v\le 1$ and hence

$$d^{\frac{p'}{4}}v^*\sum_j E_M(w_j^*w_j)vd^{\frac{p'}{4}} \;\le\; d^{\frac{p'}{2}}\,.$$

Therefore, we have

$$\sum_j\|d_j\|_2^2=\sum_j tr(E_M(d_j^*d_j)) \;=\; tr(\sum_j d^{\frac{p'}{4}}v^*E_M(w_j^*w_j)vd^{\frac{p'}{4}}) \;\le\; tr(d^{\frac{p'}{2}}) \;\le\; 1+\delta\,.$$

This yields

$$\|(w_jD^{\frac{1}{p'}})\|_{\ell_2(L_2(M))L_r(N)} \;\le\; (1+\delta)^{\frac{1}{p'}}\|\sum_j D^{\frac{1}{p'}}E_M(w_j^*w_j)D^{\frac{1}{p'}}\|^{\frac12}_{\frac{p'}{2}}\,.$$

This means that on a dense subset of $L_{p'}(N,E_M,\ell_2^c)$ we have

$$|||\,(x_j)\,||| \;=\; \inf_{x_j=y_ja}\left(\sum_k\|y_j\|_2^2\right)^{\frac12}\|a\|_r \;\le\; \|(x_j)\|_{L_{p'}(N,E_M,\ell_2^c)}\,.$$

Following the arguments in [Pis1] it is easily checked that the space $\ell_2(L_2(M))L_r(N)$ equipped with this homogenous expression $|||\ |||$ is indeed a normed space. By density we obtain the contractive inclusion

$$L_{p'}(N, E_M, \ell_2^c) \subset \ell_2(L_2(M))L_r(N) .$$

For the converse we consider $t_j = d_j d$ such that $d \in L_{2r}(N)$ and $(d_j) \subset \ell_2(L_2(M))$. Let $(x_j) \subset L_p(N, E_M; \ell_2^c)$. Then, we deduce from the Cauchy-Schwarz inequality (see [Jun1, Propostion 2.15]) that

$$\begin{aligned} |\sum_j tr(x_j^* t_j)| \ &= \ |\sum_j tr(x_j^* d_j d)| \ = \ |tr(\sum_j E_M(x_j^* d_j d))| \\ &\le \| \sum_j E_M(x_j^* y_j)\|_{r'} \|d\|_r \ \le \ \|(\sum_j E_M(x_j^* x_j))^{\frac12}\|_p \|(\sum_j E_M(d_j^* d_j))^{\frac12}\|_2 \|d\|_r \\ &= \ \|(x_j)\|_{L_p(N,E_M;\ell_2^c)} \ \|(d_j)\|_{\ell_2(L_2(M))} \|d\|_r . \end{aligned}$$

■

In the following we will keep the notation $L_p(N, E_M; \ell_2^c)$ as the closure of finite elements $(w_j D^{\frac1p})$ with respect to the norm

$$\|(w_k D^{\frac1p})\|_{L_p(N,E_M,\ell_2^c)} = \| \sum_j D^{\frac1p} E_M(w_k^* w_k) D^{\frac1p} \|_{p/2}^{1/2}$$

for all $1 \le p \le \infty$. The row version $L_p(N, E_M; \ell_2^r)$ is given by

$$\|(x_k)\|_{L_p(N,E_M;\ell_2^r)} \ = \ \|(x_k^*)\|_{L_p(N,E_M;\ell_2^c)} .$$

We will also use the free probability notation $L_p(\mathring{A})$ for elements in $L_p(\mathrm{A})$ with $E_M(x) = 0$. In the following we assume that $M \subset A_k \subset N$ are von Neumann subalgebras with a conditional expectation $E_{A_k} : N \to A_k$ satisfying $E_M E_{A_k} = E_{A_k} E_M = E_M$.

Lemma 4.2. *Let $1 \le p \le \infty$ and $X_p(c) \subset L_p(N, E_M, \ell_2^c)$ the closure of finite sequences $(x_k D^{\frac1p})$ with $x_k \in \mathring{A}$. Then $X_p(c)$ is 2-complemented in $L_p(N, E_M, \ell_2^c)$.*

Proof. Let us consider a finite sequence $(x_k D^{\frac1p})$. Then we deduce from Kadison inequality that

$$\begin{aligned} &\| \sum_k D^{\frac1p} E_M(E_{A_k}(x_k)^* E_{A_k}(x_k)) D^{\frac1p} \|_{p/2} \ \le \ \| \sum_k D^{\frac1p} E_M(E_{A_k}(x_k^* x_k)) D^{\frac1p} \|_{p/2} \\ &\le \| \sum_k D^{\frac1p} E_M(E_{A_k}(x_k^* x_k)) D^{\frac1p} \|_{p/2} \ = \ \| \sum_k D^{\frac1p} E_M(x_k^* x_k) D^{\frac1p} \|_{p/2} . \end{aligned}$$

This shows that the map $\tilde{F}((y_k)) = (E_{A_k}(y_k))$ extends to a contraction on $L_p(M, E_M, \ell_2^c)$. The same argument shows that $\tilde{E}_M((y_k)) = (E_M(y_k))$ extends to a contraction. The projection onto $X_p(c)$ is given by $(id - \tilde{E})\tilde{F}$ and has norm ≤ 2. ■

Theorem 4.3. *Let $1 < p \leq 2$, $2 \leq s < \infty$ with $\frac{1}{p} = \frac{1}{2} + \frac{1}{s}$. Let (A_k) be σ_t^ϕ-invariant algebras, faithfully independent over M. Let $x_k \in L_p(\mathring{A}_k)$. Then*

$$\begin{aligned}\frac{1}{2}\|\sum_k x_k\|_p \;\leq\; &\inf_{x_k \,=\, y_k(c)a+by_k(r)+x_k(d)} \\ &+\left(\sum_k \|x_k(d)\|_p^p\right)^{\frac{1}{p}} + \left(\sum_k \|x_k(c)\|_2^2\right)^{\frac{1}{2}} \|a\|_s + \|b\|_s \left(\sum_k \|x_k(r)\|_2^2\right)^{\frac{1}{2}} \\ &\leq Cp'\|\sum_k x_k\|_p\end{aligned}$$

where the infimum is taken over $y_k(c), y_k(r) \in L_2(\mathring{A}_k)$, $a, b \in L_s(M)$ and $x_k(d) \in L_p(\mathring{A}_k)$.

Proof. Let $1 \leq p \leq 2$ and $(x_k) \in L_p(\mathring{A}_k)$. By Lemma 1.1 and the fact that $L_p(N)$ has type p, we have

$$\|\sum_k x_k\|_p \;\leq\; 2\,\mathbb{E}\|\sum_k \varepsilon_k x_k\|_p \;\leq\; 2\left(\sum_{k=1}^n \|x_k\|_p^p\right)^{\frac{1}{p}} .$$

We denote by $X_p(d) \subset \ell_p(L_p)$ the subspace of sequences (x_k) with $x_k \in L_p(\mathring{A}_k)$. For $x_k = z_k D^{1/p'}$ with $E_M(z_k) = 0$ and $x = \sum_k z_k D^{1/p}$, we deduce from [Jun1, section 2](see also [JX1, section 7]) that

$$\|x\|_p^2 \;=\; \|x^*x\|_{\frac{p}{2}} \;\leq\; \|E_M(x^*x)\|_{\frac{p}{2}} \;=\; \|\sum_{k,j=1}^n D^{\frac{1}{p}}E_M(z_k^*z_j)D^{\frac{1}{p}}\|_{\frac{p}{2}} = \;\|\sum_{k=1}^n D^{\frac{1}{p}}E_M(z_k^*z_k)D^{\frac{1}{p}}\|_{\frac{p}{2}}.$$

By density this implies that

$$\|\sum_k x_k\|_p \;\leq\; \|(x_k)\|_{L_p(N,E_M;\ell_2^c)} .$$

whenever $(x_k) \in X_p(c)$. We define $X_p(r) = \{(x_k^*) : (x_k) \in X_p(c)\}$ with the canonical norm. By convexity we find

$$\|\sum_k x_k\|_p \;\leq\; \inf_{x_k = x_k(d)+x_k(c)+x_k(r)} \|(x_k(d))\|_{X_p(d)} + \|(x_k(c))\|_{X_p(c)} + \|(x_k(r))\|_{X_p(r)} .$$

According to (the very last and easy part) in the proof of Lemma 4.1, we obtain the first estimate. For the converse we consider $x = \sum_k x_k \in L_p(N)$ such that $x_k \in L_p(\mathring{A}_k)$. We define the functional $\Phi : X_{p'}(d) \cap X_{p'}(c) \cap X_{p'}(r) \to \mathbb{C}$ by

$$\Phi(y_k) \;=\; \sum_k tr(x_k^* y_k) .$$

The symbol $\cap$ refers to the fact that the norm in this space is given by the maximum of the three expressions. Theorem (1.2) implies that

$$\begin{aligned}|\Phi(y_k)| = |\sum_k tr(y_k^* x_k)| \ &= \ |\sum_{kj} tr(y_k^* x_j)| \\ \le \|\sum_k x_k\| \, \|\sum_j y_j\| \ &\le \ Cp' \, \|\sum_k x_k\| \, \|(y_k)\|_{X_{p'}(d)\cap X_{p'}(c) \cap X_{p'}(r)} \, .\end{aligned}$$

Here we used the orthogonality $tr(y_k^* x_j) = tr(F_j(y_k^* x_j)) = tr(E_M(y_k)^* x_j) = 0$. By the Hahn-Banach theorem, we may extend Φ to a functional $\hat{\Phi}$ on the space

$$X_p(d) \oplus_\infty X_p(c) \oplus_\infty X_p(r)$$

with the same norm. With the help of the complementation from Lemma 4.2 and Lemma 4.1, we deduce that $X_{p'}(c)^* = X_p(c)$, $X_{p'}(r)^* = X_p(r)$ and $X_{p'}(d)^* = X_p(d)$ holds with constant 2. Therefore $\hat{\Phi}$ has three components $(x_k(c)) \in X_{p'}(c)$, $(x_k(r)) \in X_{p'}(r)$ $(x_k(d)) \in X_{p'}(d)$ such that

$$\|(x_k(d))\|_{X_p(d)} + \|(x_k(c))\|_{X_p(c)} + \|(x_k(r))\|_{X_p(r)} \ \le \ 2\|\Phi\| \ \le \ 2Cp' \|\sum_k x_k\|_p \, .$$

Moreover, since $\hat{\Phi}$ extends Φ, we know that

$$(4.2)\ \ tr(x_k^* y_k) \ = \ \Phi((0,0,\cdots,0,y_k,0,\cdots)) \ = \ tr(x_k(c)^* y_k) + tr(x_k(r)^* y_k) + tr(x_k(d)^* y_k)$$

holds for all $y_k \in L_p(\AA_k)$. Since $E_M(x_k) = 0$ we deduce for all $y_k \in L_p(A_k)$ that

$$\begin{aligned} tr(x_k^* y_k) &= tr(x_k^*(y_k - E_M(y_k)) + tr(x_k^* E_M(y_k)) \\ &= tr(x_k^*(y_k - E_M(y_k)) + tr(E_M(x_k)^* E_M(y_k)) \ = \ tr(x_k^*(y_k - E_M(y_k))) \, . \end{aligned}$$

Therefore (4.2) implies indeed $x_k \ = \ x_k(d) + x_k(c) + x_k(r)$. Finally, we have to apply the second part of Lemma 4.3 and observe that we find $x_k(c) = y_k(c)a$ where $a^r \in L_1(M)$ is the density of a faithful state. Then we find $y_k(c)a = E_{A_k}(x_k(c)) = E_{A_k}(y_k(c))a$ and hence $y_k(c) = E_{A_k}(y_k(c))$. The same argument shows that $0 = E_M(x_k(c)) = E_M(y_k(c))a$ and hence $E_M(y_k(c)) = 0$. ■

Remark 4.4. *The maximal inequality of section 3 also implies an improvement for the lower estimate. Under the assumptions above, we have*

$$\begin{aligned} &\inf_{x_k = x_k(c)+x_k(r)+x_k(d)} \|(x_k(d))\|_{L_p(\ell_1)} + \|(x_k(c))\|_{L_p(N,E_M,\ell_2^c)} + \|(x_k(c))\|_{L_p(N,E_M,\ell_2^c)} \\ &\le (Cp')^{\frac{p}{2-p}} \, \|\sum_k x_k\|_p \, . \end{aligned}$$

Indeed, the modified spaces $\tilde{X}_{p'}(d) \subset L_{p'}(\ell_\infty)$ *given by sequences* (x_k) *with* $x_k \in L_{p'}(\mathrm{\AA}_k)$ *is complemented to due Doob's inequality. Then the same Hahn-Banach argument and Theorem 3.8 provide the decomposition.*

Proof of Theorem 0.1 for $1 < p \le 2$. We can not formally apply Theorem (4.3). However, we note that the right hand side of Corollary (1.5) defines space $\tilde{X}_p(c)$, $\tilde{X}_p(r)$, $\tilde{X}_p(d)$ which are complemented in $\ell_p(L_p(N, E_M; \ell_2^c))$, $\ell_p(L_p(N, E_M; \ell_2^r))$ and $\ell_p(\mathbb{N}^2; L_p(N))$ (respecting the correct switch of the indices). The Hahn-Banach theorem then provides the corresponding decomposition in combination with Lemma 4.1. ∎

Due to the recent work of Pisier/Shlyahktenko and [Jun2, Jun4], it is clear conditional expectations with respect states are very important for the theory of operators spaces and closely connected to the classical Araki-Woods factors. Let us describe an application in this direction. We consider the generators of the CAR algebra

$$a_k \;=\; 1 \otimes \cdots \otimes 1 \otimes \underbrace{e_{12}}_{k\text{-th position}} \otimes 1 \otimes \cdots \otimes 1\,.$$

Let $(\mu_k) \subset (0,1)$ and $\phi_{\mu_k} \;=\; (1-\mu_k)e_{11} + \mu_k e_{22}$. Here $e_{ij} \in M_2$ are the standard matrix units. Then the tensor product state $\phi = \otimes_{k \in \mathbb{N}} \phi_{\mu_k}$ is a quasi-free state satisfying

$$\phi_\mu(a_{i_1}^* \cdots a_{i_r}^* a_{j_1} \cdots \otimes a_{j_s}) \;=\; \delta_{rs} \prod\nolimits_{l=1}^{s} \delta_{i_l, j_l} \mu_{i_l}$$

for all increasing sequences $i_1 < \dots < i_r$ and $j_1 < \dots < j_r$. We denote by $\mathcal{N}(\mu)$ the von Neumann algebra generated by the a_k's in the GNS contruction with respect to ϕ_μ.

Theorem 4.5. *Let* D_μ *be the density of* ϕ_μ *in* $L_1(\mathcal{N}(\mu))$. *Let* $1 < p \le 2$, $\mathcal{N}$ *be a von Neumann algebra and* $x_k \in L_p(\mathcal{N})$. *Then*

$$\begin{aligned} &\Big\| \sum_k x_k \otimes D_\mu^{\frac{1}{2p}} a_k D_\mu^{\frac{1}{2p}} \Big\|_p \;\sim_{cp'} \\ &\qquad \inf_{x_k = c_k + d_k} \Big\| \Big(\sum_k (1-\mu_k)^{\frac{1}{p}} \mu_k^{\frac{1}{p'}} c_k^* c_k\Big)^{\frac{1}{2}} \Big\|_p + \Big\| \Big(\sum_k (1-\mu_k)^{\frac{1}{p'}} \mu_k^{\frac{1}{p}} d_k d_k^*\Big)^{\frac{1}{2}} \Big\|_p \end{aligned}$$

Here the infimum is taken over all (c_k), $(d_k) \subset L_p(\mathcal{N})$.

Proof. We define $q = p' \ge 2$. The natural conditional expectation $E : \mathcal{N} \otimes \mathcal{N}(\mu) \to \mathcal{N} \otimes 1$ given by $E(x \otimes y) \;=\; \phi_\mu(y)(x \otimes 1)$. We may assume that $\mathcal{N}$ is σ-finite. We define the algebras

$$\tilde{A}_k \;=\; 1 \otimes \cdots \otimes 1 \otimes \underbrace{M_2}_{k\text{-th position}} \otimes 1 \otimes \cdots$$

and $A_k = \mathcal{N} \otimes \tilde{A}_k$. It is obvious that the A_k's are (faithfully) independent over $\mathcal{N} \otimes 1$. For $q \geq 2$, we deduce from Theorem 1.2 that

$$\begin{aligned} \|\sum_k y_k^* \otimes D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}}\|_q \;\leq\; Cq \max\Bigg\{ & \Big(\sum_k \|y_k^* \otimes D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}}\|_q^q\Big)^{\frac{1}{q}}, \\ & \|\sum_k y_k y_k^* \otimes E(D_\mu^{\frac{1}{2q}} a_k D_\mu^{\frac{1}{2q}} D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}})\|_{\frac{q}{2}}^{\frac{q}{2}}, \|\sum_k y_k^* y_k \otimes E(D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}} D_\mu^{\frac{1}{2q}} a_k D_\mu^{\frac{1}{2q}})\|_{\frac{q}{2}}^{\frac{q}{2}} \Bigg\} . \end{aligned}$$

By approximation we may assume that only a finite number of the y_k's is non-zero. Thus we may consider $D_\mu = \otimes_{k=1}^m ((1-\mu_k)e_{11} + \mu_k e_{22}) \in S_1^{2^m} = (S_1^2)^{\otimes_m}$. This implies

$$\|D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}}\|_q \;=\; \|((1-\mu_k)\mu_k)^{\frac{1}{2q}} e_{21}\|_q \;=\; ((1-\mu_k)\mu_k)^{\frac{1}{2q}} .$$

It is also easy to calculate the conditional expectation

$$E(D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}} D_\mu^{\frac{1}{2q}} a_k D_\mu^{\frac{1}{2q}}) = (1-\mu_k)^{\frac{1}{q}} \mu_k^{-\frac{1}{q}} D_\mu^{\frac{1}{q}} E(a_k^* a_k) D_\mu^{\frac{1}{q}} \;=\; (1-\mu_k)^{\frac{1}{q}} \mu_k^{\frac{1}{q'}} D_\mu^{\frac{2}{q}} .$$

Similarly, we obtain

$$E(D_\mu^{\frac{1}{2q}} a_k D_\mu^{\frac{1}{2q}} D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}}) \;=\; (1-\mu_k)^{\frac{1}{q'}} \mu^{\frac{1}{q}} D_\mu^{\frac{2}{q}} .$$

Using $\big(\sum_k \|z_k\|_q^q\big)^{\frac{1}{q}} \leq \|(\sum_k z_k^* z_k)^{1/2}\|_q$, we may split the ℓ_q term according to $\mu_k \geq \frac{1}{2}$ and $1-\mu_k \geq \frac{1}{2}$ and obtain

$$\begin{aligned} & \big(\sum_k \|(1-\mu_k)^{1/2q} \mu_k^{1/2q} y_k\|_q^q\big)^{\frac{1}{q}} \\ & \leq 2^{\frac{1}{2q'} - \frac{1}{2q}} \big(\|\big(\sum_k (1-\mu_k)^{\frac{1}{q'}} \mu_k^{\frac{1}{q}} y_k y_k^*\big)^{\frac{1}{2}}\|_q + \|\big(\sum_k (1-\mu_k)^{\frac{1}{q}} \mu_k^{\frac{1}{q'}} y_k^* y_k\big)^{\frac{1}{2}}\|_q \big) . \end{aligned}$$

Therefore the Rosenthal inequality in this case simplifies to

$$\begin{aligned} & \|\sum_k y_k^* \otimes D_\mu^{\frac{1}{2q}} a_k^* D_\mu^{\frac{1}{2q}}\|_q \\ & \leq\; 4Cq \max\big\{ \|\big(\sum_k (1-\mu_k)^{\frac{1}{q'}} \mu_k^{\frac{1}{q}} y_k y_k^*\big)^{\frac{1}{2}}\|_q, \|\big(\sum_k (1-\mu_k)^{\frac{1}{q}} \mu_k^{\frac{1}{q'}} y_k^* y_k\big)^{\frac{1}{2}}\|_q \big\}. \end{aligned}$$

Now, we observe that

$$tr(D_\mu^{\frac{1}{2q'}} a_j^* D_\mu^{\frac{1}{2q'}} D_\mu^{\frac{1}{2q}} a_k D_\mu^{\frac{1}{2q}}) \;=\; \delta_{k,j} (1-\mu_k)^{\frac{1}{2}} \mu_k^{-\frac{1}{2}} \phi_\mu(a_k^* a_k) \;=\; \delta_{kj} \sqrt{(1-\mu_k)\mu_k} .$$

Given $x_k \in L_p(\mathcal{N})$ we deduce

$$\left| \sum_k tr_{\mathcal{N}}(y_k^* x_k) \sqrt{(1-\mu_k)\mu_k} \right| \;=\; \left| tr_{\mathcal{N} \otimes \mathcal{N}(\mu)} \left((\sum_j y_j^* \otimes D_\mu^{\frac{1}{2q}} a_j^* D_\mu^{\frac{1}{2q}})(\sum_k x_k \otimes D_\mu^{\frac{1}{2p}} a_k D_\mu^{\frac{1}{2p}}) \right) \right|$$

$$\leq \|\sum_j y_j^* \otimes D_\mu^{\frac{1}{2q}} a_j^* D_\mu^{\frac{1}{2q}}\|_q \, \|\sum_k x_k \otimes D_\mu^{\frac{1}{2p}} a_k D_\mu^{\frac{1}{2p}}\|_p \;\leq\; 4Cq\|\sum_k x_k \otimes D_\mu^{\frac{1}{2p}} a_k D_\mu^{\frac{1}{2p}}\|_p$$

$$\max\{\|(\sum_k (1-\mu_k)^{\frac{1}{p}} \mu_k^{\frac{1}{q}} y_k y_k^*)^{\frac{1}{2}}\|_q, \|(\sum_k (1-\mu_k)^{\frac{1}{q}} \mu_k^{\frac{1}{p}} y_k^* y_k)^{\frac{1}{2}}\|_q\} \,.$$

By duality this implies the decomposition. The converse inequality follows as in Theorem 4.3. ■

Remark 4.6. (1) According to [Jun4] this inequality also holds for $p = 1$. We conjecture that there is is continuous passage for $p \to 1$ improving the constant above.

(2) The same proof works for q-Araki Wood factors defined by Shlyahktenko for $q = 0$ and for $q \in (-1, 1)$ by Hiai [Hia]. For $q = 1$ we refer to [Jun4] for the appropriate gaussian substitute.

(3) Using this Khintchine type inequality 4.5, we can deduce, as in [Jun4], that every quotient of $R_p \oplus C_p$ completely embeds into $L_p(R_{III_1})$ the L_p-spaces associated to the hyperfinite III_1 factor. In particular, the operator space OH embeds into $L_p(\mathcal{N})$ where $\mathcal{N}$ is a hyperfinite factor of type III_λ, $0 < \lambda \leq 1$. We refer to [Xu] for more details.

Remark 4.7. There is also a version of the Rosenthal inequality for the non-faithful case in the range $1 < p \leq 2$. We assume that the algebras (A_k) are independent over $E_M : N \to M$, the support of E is e and ϕ is faithful on eNe. The main technical difference is that we have to introduce two extra spaces

$$X_p(se) \;=\; \{\sum_k s_k x_k e \;:\; x_k \in L_p(A_k)\} \quad \text{and} \quad X_p(es) \;=\; \{\sum_k e x_k s \;:\; x_k \in L_p(A_k)\}$$

It is easily shown that that $X_p(se)$ and $X_p(es)$ are complemented in

$$X_p \;=\; \{\sum_k x_k \;:\; x_k \in L_p(\mathring{A}_k)\} \,.$$

Thanks to Lemma 2.1, we are able to describe the (anti-linear) dual $X_{p'}(se)$ of $X_p(se)$ as an intersection of two terms, an $\ell_{p'}$-term and a column square function. Using the duality argument from Theorem 4.3 this yields

$$\|\sum_k s_k x_k e\|_p \leq \inf_{s_k x_k e = s_k x_k(c) e + s_k x_k(d) e} \|(\sum_k E_M(x_k(c)^* s_k x_k(c)))^{\frac{1}{2}}\|_p + \left(\sum_k \|s_k x_k(d) e\|_p^p\right)^{\frac{1}{p}}$$
$$\leq C\sqrt{p'}\|\sum_k s_k x_k e\|_p \,.$$

A similar result holds for $X_p(es)$ and Theorem 4.3 applies to the faithful part: Let $x_k \in L_p(A_k)$ and $y_k = x_k - E_M(x_k)$. First we have

$$\|\sum_k x_k\|_p \sim_5$$

$$\max\left\{\|\sum_k E_M(x_k)\|_p, \left(\sum_k \|s_k x_k s_k\|_p^p\right)^{\frac{1}{p}}, \|\sum_k e y_k e\|_p, \|\sum_k s_k x_k e\|_p, \|\sum_k e x_k s_k\|_p\right\}.$$

According to Theorem (4.3) we have that

$$\|\sum_k e y_k e\|_p \sim_{Cp'} \inf_{y_k \,=\, y_k(c)+y_k(r)+y_k(d)}$$

$$\|(e y_k(c) e)\|_{L_p(N,E_M,\ell_2^c)} + \|(e y_k(r) e)\|_{L_p(N,E_M,\ell_2^r)} + \|(e y_k(d) e)\|_{\ell_p(L_p))} \, .$$

Here the infimum is taken over $y_k(c), y_k(r), y_k(d) \in L_p(\mathring{A}_k)$. This complicated expression involving maximum and infimum is particulary interesting in the connection with independent copies (as in [Jun4]). In this case, the expressions are symmetric. This formula can be used to prove that subsymmetric sequences in $L_p(N)$, $1 < p \leq 2$ are symmetric (see [JR] for more details).

5. Symmetric subspaces of L_p

In this section, we will prove a quantitative version of the Kadec-Pełczyński alternative, the noncommutative analogue of the corresponding result in [JMST]. It will be convenient to state these result in the Banach space and operator space setting. We refer to [Pis2] for a general background on operator spaces and completely bounded maps and to [JNRX] for the operator space structure of noncommutative L_p spaces. In this paper we focus on subspaces X, Y of $L_p(N)$. In this situation the cb-norm of a linear map $T : X \to Y$ is given by

$$\|T\|_{cb} \;=\; \|id_{L_p(B(\ell_2))} \otimes T : L_p(B(\ell_2); X) \to L_p(B(\ell_2); Y)\| \, .$$

Here $L_p(B(\ell_2); X) \subset L_p(B(\ell_2) \otimes N)$ consist of the matrices with coefficients in X. In other words, the cb-norm is calculated with matrix valued coefficients instead of scalar valued coefficients. Note that martingale inequalities often automatically extend to the matrix valued case.

Lemma 5.1. *Let N be a hyperfinite type III_λ factor where $0 \leq \lambda \leq 1$. Let $1 < p < \infty$, then $L_p(N)$ has a completely unconditional finite dimensional decomposition (see below for a definition).*

Proof. In the range $0 < \lambda \leq 1$, we may assume that N is an ITPFI factor. In general (including $\lambda = 0$), we can always find a normal faithful state ϕ, and an increasing sequence of finite subalgebras N_n with ϕ-invariant conditional expectation $E_n : N \to N_n$ see [JRX1]. We define the difference operators $d_n = E_n - E_{n-1}$ where $E_0 = 0$. Note that the spaces $F_n = d_n(N)$ are finite dimensional and every element can be written uniquely as $x = \sum_n d_n(x)$. Thus $L_p(N)$ has a finite dimensional decomposition. Such a decomposition is called completely unconditional if all the maps $T_\varepsilon(\sum_n d_n) = \sum_n \varepsilon_n d_n$ with $\varepsilon_n = \pm 1$ are uniformly completely bounded. This means that the maps $id_{L_p(B(\ell_2))} \otimes T_\varepsilon$ are uniformly bounded, i.e. there exists a constant $c(p)$ such that

$$\| \sum_n \varepsilon_n (id \otimes d_n)(x) \|_p \leq c(p) \| \sum_n (id \otimes d_n)(x) \|_p \, . \tag{5.1}$$

holds for all choices of signs (ε_n) and $x \in L_p(B(\ell_2) \otimes N)$. Equation (5.1) is a direct consequence of the Burkholder-Gundy inequalities [PX1, JX1]. ■

Theorem 5.2. *Let N be a hyperfinite von Neumann algebra. Let $2 < p < \infty$ and $(x_k) \subset L_p(N)$ be a sequence of norm one vectors, which converges weakly to 0. Then there exist constants $0 \leq c_1, c_2 \leq 1$, a subsequence $(\tilde{x}_n)$ of (x_n) such that*

$$\| \sum_n a_n \otimes \tilde{x}_n \|_p \sim_{c(p)} \left(\sum_n \|a_n\|_p^p \right)^{\frac{1}{p}} + c_1 \| (\sum_n a_n^* a_n)^{\frac{1}{2}} \|_p + c_2 \| (\sum_n a_n a_n^*)^{\frac{1}{2}} \|_p$$

holds for all finitely supported sequence $(a_n) \subset S_p$.

Proof. Using Remark 1.4 we may assume that N_* is separable. We will first use a standard trick in order to ensure that we may work with a factor. Indeed, let ϕ be a normal faithful state. We consider the crossed product $M = \otimes_{n \in \mathbb{N}} (N, \phi) \rtimes G$ between the infinite tensor product $\otimes_{n \in \mathbb{N}} (N, \phi)$ and the discrete group G of all finite permutations on $\mathbb{N}$. Any finite permutation acts on the infinite tensor product by shuffling the corresponding coordinates. Clearly, we also have a conditional expectation $E_0 : M \to N$ obtained by first projecting onto the identity element e in G and then to the first component in the infinite tensor product. According to [HW, Proof of Theorem 2.6] we know that M is a hyperfinite factor. According to Connes' characterization M is type III_λ for some $0 \leq \lambda \leq 1$ [Con]. According to Lemma 5.1, we have a normal faithful state, conditional expectations $E_k : N \to N_k$ onto finite dimensional subalgebras N_k. Now, the proof follows very closely its commutative model see [JMST, Theorem 1.14,p=50]. Using the gliding hump procedure, we may find a perturbation of a subsequence $(\hat{x}_n)$ and a subsequence $\hat{E}_k$ such that

i) $\hat{E}_n(\hat{x}_n) = \hat{x}_n$,
ii) $\hat{E}_n(\hat{x}_k) = 0$ for all $k > n$,
iii) $\lim_k \hat{E}_n(\hat{x}_k^* \hat{x}_k) = y_n$ and $\|\hat{E}_n(\hat{x}_k^* \hat{x}_k) - y_n\|_{\frac{p}{2}} \le \varepsilon 2^{-k}$ for $k > n$,
iv) $\lim_k \hat{E}_n(\hat{x}_k \hat{x}_k^*) = z_n$ and $\|\hat{E}_n(\hat{x}_k \hat{x}_k^*) - z_n\|_{\frac{p}{2}} \le \varepsilon 2^{-k}$ for $k > n$.

Here $\varepsilon > 0$ is arbitrary and will be chosen after knowing the y_n's. It follows immediately from iii) that (y_n) is a martingale, namely $\hat{E}_n(y_{n+1}) = y_n$. Since $1 < \frac{p}{2}$, we deduce from the Burkholder-Gundy inequalities [JX1] that (y_n) is convergent to some $y \in L_p(M)$. Similarly, we obtain that (z_n) is convergent to some $z \in L_p(N)$. We define $c_1 = \|y\|_{p/2}^{1/2}$ and $c_2 = \|z\|_{p/2}^{1/2}$. Passing to another subsequence denoted by $(\tilde{x}_n)$ $(\tilde{E}_n)$, we may assume

$$\|\tilde{E}_n(\tilde{x}_{n+1}^* \tilde{x}_{n+1}) - y\|_{\frac{p}{2}} \le 2^{-(n+2)} \|y\|_{\frac{p}{2}} \quad \text{and} \quad \|\tilde{E}_n(\tilde{x}_{n+1} \tilde{x}_{n+1}^*) - z\|_{\frac{p}{2}} \le 2^{-(n+2)} \|z\|_{\frac{p}{2}} .$$

We apply Burkholder's inequality [JX1] and find

$$\begin{aligned} \|\sum_n a_n \otimes \tilde{x}_n\|_p \sim_{c(p)} & \left(\sum_n \|a_n \otimes \tilde{x}_n\|_p^p \right)^{\frac{1}{p}} \\ & + \|\sum_n a_n^* a_n \otimes \tilde{E}_n(\tilde{x}_{n+1}^* \tilde{x}_{n+1})\|_{\frac{p}{2}}^{\frac{1}{2}} + \|\sum_n a_n^* a_n \otimes \tilde{E}_n(\tilde{x}_{n+1} \tilde{x}_{n+1}^*)\|_{\frac{p}{2}}^{\frac{1}{2}} . \end{aligned}$$

From perturbation we have $\frac{1}{2} \le \|\tilde{x}_n\| \le 2$. The triangle inequality implies

$$\begin{aligned} \|\sum_n a_n^* a_n \otimes \tilde{E}_n(\tilde{x}_{n+1}^* \tilde{x}_{n+1}) - \sum_n a_n^* a_n \otimes y\|_{\frac{p}{2}} & \le \sum_n \|a_n^* a_n\| \|\tilde{E}_n(\tilde{x}_{n+1}^* \tilde{x}_{n+1}) - y\|_{\frac{p}{2}} \\ \le \frac{1}{2} \|y\|_{\frac{p}{2}} \sup_n \|a_n^* a_n\|_{\frac{p}{2}} & \le \frac{1}{2} c_1^2 \|\sum_n a_n^* a_n\|_{\frac{p}{2}} . \end{aligned}$$

Therefore, we get

$$\begin{aligned} c_1^2 \|\sum_n a_n^* a_n\|_{\frac{p}{2}} &= \|\sum_n a_n^* a_n \otimes y\|_{\frac{p}{2}} \\ &\le \|\sum_n a_n^* a_n \otimes \tilde{E}_n(\tilde{x}_{n+1}^* \tilde{x}_{n+1})\| + \|\sum_n a_n^* a_n \otimes \tilde{E}_n(\tilde{x}_{n+1}^* \tilde{x}_{n+1}) - \sum_n a_n^* a_n \otimes y\|_{\frac{p}{2}} \\ &\le \frac{3}{2} c_1^2 \|\sum_n a_n^* a_n\| . \end{aligned}$$

The same argument applies to the last term and the assertion follows. ■

Let us recall some notation from the theory of operator spaces. The spaces C_p and R_p are defined as subspaces of S_p given by

$$C_p \;=\; \overline{\mathrm{span}}\{e_{k1} \,:\, k\in\mathbb{N}\} \quad \text{and} \quad R_p \;=\; \overline{\mathrm{span}}\{e_{1k} \,:\, k\in\mathbb{N}\}\,.$$

As an application, we obtain an operator space version of the Kadec-Pełzsiński alternative.

Corollary 5.3. *Let N be hyperfinite. Let $2\le p<\infty$ and (x_n) be a sequence which converges to 0 weakly. Then (x_n) contains a subsequence (x_n') which is completely equivalent to ℓ_p, R_p, C_p or $R_p\cap C_p$.*

Proof. Let (x_n') the subsequence from Theorem 5.2. If $c_1=c_2=0$, then (x_n') is completely equivalent to the unit vector basis of ℓ_p. If $c_1=0$ and $c_2>0$, then we find a copy of R_p. Similarly, if $c_2=0$ and $c_1>0$ it turns out to be C_p. The case $c_1>0$ and $c_2>0$ yields $R_p\cap C_p$. ■

Remark 5.4. We deduce in particular that every infinite dimensional subspace $X\subset L_p(N)$ contains a completely symmetric subspace, i.e. a basic sequence (x_k) such that

$$\|\sum_k \varepsilon_k a_{\pi(k)}\otimes x_k\| \;\le\; C\|\sum_k a_k\otimes e_k\| \tag{5.2}$$

holds for all $a_k\in L_p(M)$, $\varepsilon_k=\pm1$ and permutations π of the integers. This problem (even for scalar coefficients) is open for $1\le p<2$. The problem is also open for $2<p<\infty$ without assuming that N is hyperfinite. On the Banach space level we refer to [RX] and [Ran] for different versions of the Kadec-Pełcziński alternative.

We will now show that conversely the only symmetric subspaces of $L_p(N)$ are the one found in (5.3). The next result is a our starting point.

Theorem 5.5. *Let $2\le p<\infty$, N a von Neumann algebra and $x_{ij}\in L_p(N)$. Then*

$$\left(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_{i\pi(i)}\|_p^p\right)^{\frac1p} \;\sim_{c_p}\; \left(\frac1n\sum_{i,j=1}^n \|x_{ij}\|_p^p\right)^{\frac1p} + \left\|\left(\frac1n\sum_{i,j=1}^n (x_{ij}^*x_{ij}+x_{ij}x_{ij}^*)\right)^{\frac12}\right\|_p .$$

Here the expectation is taken over all choices of sign $\varepsilon_i=\pm1$ and all permutation π on $\{1,..,n\}$.

Proof. By approximation, we may assume that N is σ-finite and ψ is normal faithful state. We consider $\Omega=\{-1,1\}^n\times\Pi_n$, where Π_n is the set of all permutations on $\{1,...,n\}$. The Haar measure on this group is the product measure $\mu=\varepsilon\otimes\nu$ of the normalized counting measures ε and ν on $\{-1,1\}^n$, Π_n, respectively. The underlying von Neumann algebra is then given by $\mathcal{N} = L_\infty(\Omega, 2^\Omega, \mu) \otimes N$ with respect to the state

$$\phi(x) \;=\; \int_\Omega \psi(x(\omega))\, d\mu(\omega)\;.$$

In order to apply Burkholder's inequality we have to use the right filtrations which we have taken from [JMST]. For $k = 1, ..., n$ we consider the functions $f_k : \Pi_n \to \mathbb{R}$, $f_k(\pi) = \pi(k)$. The σ-algebra Σ_k^2 is defined as the smallest σ-algebra on Π_n making $f_1, ..., f_k$ measurable. By Σ_k^1 we denote the smallest σ-algebra on $\{-1,1\}^n$ making $\varepsilon_1, ..., \varepsilon_k$ measurable. Let Σ_k be the σ-algebra generated by $\Sigma_k^1 \times \Sigma_k^2$. We find the martingale filtration $(\mathcal{N}_k)_k$ where

$$\mathcal{N}_k \;=\; L_\infty(\Omega, \Sigma_k, \mu) \otimes N\;.$$

We denote by E_k the conditional expectation on $L_\infty(\Omega, \Sigma_k, \mu)$. Then $\mathcal{E}_k = E_k \otimes id$ is a ϕ-invariant conditional expectation onto $\mathcal{N}_k$. After these preliminaries we consider

$$X \;=\; \sum_{i=1}^n \varepsilon_i x_{i\pi(i)} \in L_p(\mathcal{N})\;.$$

We assume in addition $x_{kj} = 0$ for $k > \frac{n}{2}$. The upper estimate for general matrices (x_{ij}) then follows from the triangle inequality. We note that $d_k = \varepsilon_k x_{k\pi(k)}$ lies in $\mathcal{N}_k$ and

$$\mathcal{E}_{k-1}(\varepsilon_k x_{k\pi(k)}) \;=\; \mathbb{E}(\varepsilon_k)\mathcal{E}_{k-1}(x_{k\pi(k)}) \;=\; 0\;.$$

Therefore, Burkholder's inequality [JX1] implies

$$\|X\|_p \;\le\; c_p \left(\sum_{k=1}^n \|x_{k\pi(k)}\|_p^p\right)^{\frac{1}{p}} + \|\sum_{k=1}^n \mathcal{E}_{k-1}(x_{k\pi(k)}^* x_{k\pi(k)} + x_{k\pi(k)} x_{k\pi(k)}^*)\|_{\frac{p}{2}}^{\frac{1}{2}}\;.$$

Clearly, for all $k = 1, .., n$, we have

$$\|\varepsilon_k x_{k\pi(k)}\|_p^p \;=\; \sum_{j=1}^n \mathrm{Prob}(\pi(k) = j)\|x_{kj}\|_p^p \;=\; \sum_{j=1}^n \frac{(n-1)!}{n!}\|x_{kj}\|_p^p \;=\; \frac{1}{n}\sum_{j=1}^n \|x_{kj}\|_p^p\;.$$

Hence, we get

$$\left(\sum_{k=1}^n \|x_{k\pi(k)}\|_p^p\right)^{\frac{1}{p}} \;=\; \left(\frac{1}{n}\sum_{kj=1}^n \|x_{kj}\|_p^p\right)^{\frac{1}{p}}\;.$$

Let E_{k-1}^2 be the conditional expectation onto $L_\infty(\Pi_n, \Sigma_k^2, \mu)$. We observe that

$$\mathcal{E}_{k-1}(x_{k\pi(k)}^* x_{k\pi(k)}) \;=\; (E_{k-1}^2 \otimes id)(x_{k\pi(k)}^* x_{k\pi(k)})\;.$$

The atoms in Σ_{k-1}^2 are indexed by a tuples $(i_1, ..., i_{k-1})$ such that the i_j's are mutually different numbers in $\{1, .., n\}$. More precisely,

$$A_{(i_1,\dots,i_{k-1})} \;=\; \{\pi \,|\, \pi(1) = i_1, \cdots \pi(k-1) = i_{k-1}\}\;.$$

Clearly, the cardinality of $A_{(i_1,...,i_{k-1})}$ is the cardinality of $\Pi_{n-(k-1)}$, i.e. $(n-k+1)!$. Therefore letting $\alpha_k = \frac{(n-k+1)!}{n!}$, we get

$$(E^2_{k-1} \otimes id)(x^*_{k\pi(k)} x_{k\pi(k)}) \;=\; \sum_{(i_1,...,i_k)} 1_{A_{(i_1,...,i_{k-1})}}(\pi) \alpha_k^{-1} \int_{A_{(i_1,...,i_{k-1})}} x^*_{k\pi(k)} x_{k\pi(k)} \, d\nu(\pi) \, .$$

For fixed $(i_1, .., i_{k-1})$, we consider $B = \{i_1, ..., i_{k-1}\}$ and deduce for $k \le \frac{n}{2}$

$$\alpha_k^{-1} \int_{A_{(i_1,...,i_{k-1})}} x^*_{k\pi(k)} x_{k\pi(k)} \, d\nu(\pi) \;=\; \frac{1}{n-k+1} \sum_{j \notin B} x^*_{kj} x_{kj} \;\le\; \frac{2}{n} \sum_{j=1}^{n} x^*_{kj} x_{kj} \, .$$

Hence for all $k \le \frac{n}{2}$ we have

$$(E^2_{k-1} \otimes id)(x^*_{k\pi(k)} x_{k\pi(k)}) \;\le\; \frac{2}{n} \sum_{j=1}^{n} x^*_{kj} x_{kj} \, .$$

Therefore $x_{kj} = 0$ for $k \ge \frac{n}{2}$ implies

$$\sum_{k=1}^{n} \mathcal{E}_{k-1}(x^*_{k\pi(k)} x_{k\pi(k)}) \;\le\; \frac{2}{n} \sum_{kj} x^*_{kj} x_{kj}$$

for all π. The same argument applies for $x_{k\pi(k)} x^*_{k\pi(k)}$. Therefore the upper estimate follows from the triangle inequality. For the lower estimate we use $\frac{p}{2} \ge 1$ and the orthogonality of the Rademacher variables:

$$\|X\|_p^2 = (\mathbb{E}\|X^*X\|_{\frac{p}{2}})^{\frac{2}{p}} \;\ge\; \|\mathbb{E}(X^*X)\|_{\frac{p}{2}} \;=\; \|\sum_{k=1}^{n} \int_{\Pi_n} x^*_{k\pi(k)} x_{k\pi(k)}\|_{\frac{p}{2}} \;=\; \|\sum_{kj=1}^{n} \frac{1}{n} x^*_{kj} x_{kj}\|_{\frac{p}{2}} \, .$$

The same calculation involving XX^* yields the other square function estimate. Since $L_p(\mathcal{N})$ has cotype p we trivially find the missing ℓ_p-estimate. ■

Remark 5.6. The same argument applies for subgroups $G = G_1 \times \cdots \times G_m \subset \Pi_n$ where G_i is the permutation group of an interval $I_i \subset \{1, ..., n\}$ of cardinality n_i . If the intervals I_i are disjoint, we obtain

$$\left(\mathbb{E}\| \sum_{i=1}^{n} \varepsilon_i x_{i\pi(i)} \|_p^p \right)^{\frac{1}{p}}$$
$$\sim_{c_p} \quad \left(\sum_{i=1}^{m} \frac{1}{n_i} \sum_{k,l \in I_i} \|x_{kl}\|_p^p \right)^{\frac{1}{p}} + \| \left(\sum_{i=1}^{m} \sum_{k,l \in I_i} \frac{1}{n_i} \sum_{kl \in I_i} (x^*_{kl} x_{kl} + x_{kl} x^*_{kl}) \right)^{\frac{1}{2}} \|_p \, .$$

A symmetric space is a Banach space X with a (finite or infinite) basis (e_k) such that for all $(\alpha_k) \subset \mathbb{C}$, all signs ε_k and all permutations π

$$\| \sum_k \varepsilon_k \alpha_{\pi(k)} e_k \|_X \;\le\; C \,\| \sum_k \alpha_k e_k \|_X \;.$$

Let us denote $Sym(X)$ the infimum over all constant C such that there exists a basis with this property. The well-known space $J_p(a,b)$ is a weighted intersection of ℓ_p and ℓ_2.

$$\|x\|_{J_p(a,b)} \;=\; \max\{a\|x\|_p, b\|x\|_2\} \;.$$

We recall the Banach-Mazur distance

$$d(E,F) \;=\; \inf_{T:E\to F \text{isomorphism}} \|T\|\|T^{-1}\| \;.$$

Theorem 5.5 implies the noncommutative analogue of [JMST, Theorem 1.1]:

Corollary 5.7. *Let $2 \le p < \infty$ and X be an n dimensional subspace of $L_p(N)$. Then there exists a, b such that*

$$d(J_p(a,b), X) \;\le\; c_p \, Sym(X)^2 \;.$$

Proof. Let $e_1, ..., e_n \in L_p(N)$ be a symmetric basis in X, then for all coefficients $\alpha_1, ..., \alpha_n$, we have

$$\frac{1}{C} \| \sum_{k=1}^n \alpha_k e_k \|_X \;\;\le\;\; \left(\mathbb{E} \| \sum_{k=1}^n \varepsilon_k \alpha_k e_{\pi(k)} \|_p^p \right)^{\frac{1}{p}} \;\le\; C \,\| \sum_{k=1}^n \alpha_k e_k \|_X \;.$$

According to Theorem 5.5, we deduce

$$\left(\mathbb{E} \| \sum_{k=1}^n \varepsilon_k \alpha_k e_{\pi(k)} \|_p^p \right)^{\frac{1}{p}} \;\;\sim_{c_p}\;\; \max \left\{ a \left(\sum_{k=1}^n |\alpha_k|^p \right)^{\frac{1}{p}}, b \left(\sum_{k=1}^n |\alpha_k|^2 \right)^{\frac{1}{2}} \right\} \;,$$

where

$$a = a(e_1, .., e_n) \;=\; \left(\frac{1}{n} \sum_{k=1}^n \|e_k\|_p^p \right)^{\frac{1}{p}} \;\text{ and }\; b \;=\; b(e_1, .., e_n) \;=\; \| \left(\frac{1}{n} \sum_{k=1}^n (e_k^* e_k + e_k e_k^*) \right)^{\frac{1}{2}} \|_p \;.$$

We can always assume $\|e_k\|_p = 1$ for all k and deduce $0 \le a, b \le 1$. Taking the infimum over all normalized symmetric bases, we may find a sequence $(e_1(j), ..., e_n(j))_{j\in\mathbb{N}}$ such that the corresponding parameters $a(j)$ and $b(j)$ converge to a, b, respectively. Using $d(J_p(a,b), J_p(a',b')) \;\le\; \max\{\frac{a}{a'}, \frac{b}{b'}\} \max\{\frac{a'}{a}, \frac{b'}{b}\}$ we deduce the assertion. ∎

Proof of Theorem 0.2. We define

$$a \;=\; \frac{1}{n}\left(\sum_{k=1}^{n}\|e_k\|_p^p\right)^{\frac{1}{2}} \;,\; c \;=\; \|(\frac{1}{n}\sum_{k=1}^{n}e_k^*e_k)^{\frac{1}{2}}\|_p \quad\text{and}\quad r \;=\; \|(\frac{1}{n}\sum_{k=1}^{n}e_ke_k^*)^{\frac{1}{2}}\|_p \;.$$

Then Theorem 5.5 implies (0.4). The second assertion is a standard reformulation from the theory of operator spaces (see the following Remark (5.8) below). ∎

Remark 5.8. In the theory of operator space the analogue of the Banach Mazur distance is defined as $d_{cb}(X,Y) = \inf \|T\|_{cb}\|T^{-1}\|_{cb}$ (see [Pis2] for details). Therefore equation (0.4) implies that

$$d_{cb}(\mathrm{span}\{e_k : 1\le k\le n\}, a\ell_p^n \cap cC_p^n \cap rR_p^n) \;\le\; c_0p^2C^2 \;.$$

Corollary 5.9. *Let $2\le p<\infty$ and $X\subset L_p(N)$ an infinite dimensional subspace with a symmetric basis, then X is isomorphic to ℓ_p or ℓ_2. If X has a completely symmetric basis, i.e.* (5.2) *is satisfied for all $m\in\mathbb{N}$ and $a_k\in L_p(M_m)$, then X is completely isomorphic to ℓ_p, R_p, C_p or $R_p\cap C_p$.*

Proof. Let $(e_k)\subset X$ be a (completely) symmetric basis. For fixed $n\in\mathbb{N}$, we consider

$$a_n \;=\; \frac{1}{n}\left(\sum_{k=1}^{n}\|e_k\|_p^p\right)^{\frac{1}{2}} \;,\; c_n \;=\; \|(\frac{1}{n}\sum_{k=1}^{n}e_k^*e_k)^{\frac{1}{2}}\|_p \quad\text{and}\quad r_n \;=\; \|(\frac{1}{n}\sum_{k=1}^{n}e_ke_k^*)^{\frac{1}{2}}\|_p \;.$$

Note that $\sup_n a_n$, $\sup_n c_n$ and $\sup_n r_n$ can be estimated by $\sup_k\|e_k\|$. Passing to a subsequence, we assume that the three sequences are converging to a, c and r. We deduce that for any finitely supported sequence (a_k) we have

$$\|\sum_k a_k\otimes e_k\|_p \sim_{C^2c_p} a\left(\sum_{k=1}^{n}\|a_k\|_p^p\right)^{\frac{1}{p}} + b\|(\sum_{k=1}^{n}a_k^*a_k)^{\frac{1}{2}}\|_p + c\|(\sum_{k=1}^{n}a_ka_k^*)^{\frac{1}{2}}\|_p \;.$$

In the Banach space case this only holds for scalar sequences and then X is isomorphic to ℓ_p if $c=0=r$ and to ℓ_2 if $\max\{c,r\}>0$. In the operator space case, we obtain a complete isomorphism with ℓ_p if $c=0=r$. If $r=0$, we obtain C_p. If $c=0$, we get R_p. Finally $R_p\cap C_p$ occurs if c and r are both positive. ∎

Remark 5.10. In [JR] we will show that every subsymmetric basic sequence is symmetric. A sequence (e_k) is called subsymmetric if

$$\|\sum_k \varepsilon_k a_k\otimes e_{j_k}\|_p \sim_C \|\sum_k a_k\otimes e_k\|$$

holds for all increasing sequences (j_k) of integers. Therefore Corollary 0.3 yields a characterization of spaces with a subsymmetric basis.

Corollary 5.11. *Let $2 < p < \infty$ and N be a finite von Neumann algebra. Then there is no sequence $(e_k) \subset L_p(N)$ such that*

$$\|\sum_k a_k \otimes e_k\| \sim_C \|(\sum_k a_k^* a_k)^{\frac{1}{2}}\|_p \tag{5.3}$$

holds for some constant C and all $m \in \mathbb{N}$ and $(a_k) \subset L_p(M_m)$. In particular, C_p and R_p do not completely embed into $L_p(N)$.

Proof. Let us assume τ is a normal faithful trace on N with $\tau(1) = 1$. Let $(e_k) \subset L_p(N,\tau)$ and $C > 0$ such that (5.3) holds. Then we see that

$$\|[a_{ik}]\|_{S_p} \sim_C \|\sum_{ik} a_{ik} e_{1,i} \otimes e_k\|_{L_p(B(\ell_2)\bar{\otimes}N)}$$

holds for any matrix $[a_{ik}]$. Note that for $x = \sum_{ik} a_{ik} \otimes e_{1,i} \otimes e_k$ and $x_i = \sum_k a_{ik} e_k$ Hölder's inequality implies

$$\|x\|_2^2 = \|xx^*\|_1 \;=\; \|\sum_i x_i x_i^*\|_{L_1(N,\tau)} \;\le\; \|\sum_i x_i x_i^*\|_{\frac{p}{2}} \;=\; \|x\|_p^2\,.$$

This tells us that on the subspace $Y = \mathrm{span}\{e_{1,i} \otimes e_k\}$ the norm in $L_p \cap L_2$ and L_p coincide. Thus we have found an embedding of S_p in $L_p(B(\ell_2)\bar{\otimes}N) \cap L_2(B(\ell_2)\bar{\otimes}N)$. According to [Jun3] the latter space embeds into $L_p(M)$ for a finite von Neumann algebra M. Thus we obtain an embedding of S_p into $L_p(M)$. This is absurd in view of the results in [Suk]. The proof for R_p is identically the same. ■

6. Unitary ideals embedding in $L_p(N)$ for $p \ge 2$

In this section we will characterize those unitary ideals which can be embedded into some noncommutative L_p space for $2 \le p < \infty$. In a preliminary step we consider spaces with a bisymmetric basis:

Proposition 6.1. *Let $2 \le p < \infty$ and $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras and $n \in \mathbb{N}$. Let $x_{ij} \in L_p(\mathcal{N})$ and $y_{ij} \in L_p(\mathcal{M})$ be two $n \times n$ matrices. Then there exist constants $c_1, ..., c_9$ depending on the matrix x_{ij} such that*

$$\left(\mathbb{E}\|\sum_{i,j=1}^n \varepsilon_i \varepsilon'_j x_{\pi(i),\pi'(j)} \otimes y_{ij}\|_p^p\right)^{\frac{1}{p}} \sim_{c_p}$$

$$c_1\left(\sum_{i,j=1}^n \|y_{ij}\|_p^p\right)^{\frac{1}{p}} + c_2\left(\sum_{j=1}^n \|\left(\sum_{i=1}^n y_{ij}^* y_{ij}\right)^{\frac{1}{2}}\|_p^p\right)^{\frac{1}{p}} + c_3\left(\sum_{j=1}^n \|\left(\sum_{i=1}^n y_{ij} y_{ij}^*\right)^{\frac{1}{2}}\|_p^p\right)^{\frac{1}{p}}$$

$$+ c_4 \left(\sum_{i=1}^n \| \left(\sum_{j=1}^n y_{ij}^* y_{ij} \right)^{\frac{1}{2}} \|_p^p \right)^{\frac{1}{p}} + c_5 \left(\sum_{i=1}^n \| \left(\sum_{j=1}^n y_{ij} y_{ij}^* \right)^{\frac{1}{2}} \|_p^p \right)^{\frac{1}{p}}$$

$$+ c_6 \| \left(\sum_{i,j=1}^n y_{ij}^* y_{ij} \right)^{\frac{1}{2}} \|_p + c_7 \| \left(\sum_{i,j=1}^n y_{ij} y_{ij}^* \right)^{\frac{1}{2}} \|_p$$

$$+ c_8 \| \sum_{i,j=1}^n e_{ji} \otimes y_{ij} \|_{L_p(M_n \otimes \mathcal{M})} + c_9 \| \sum_{i,j=1}^n e_{ij} \otimes y_{ij} \|_{L_p(M_n \otimes \mathcal{M})} \,.$$

Here the expectation is taken over independent copies ε_i, ε_i' of Rademacher variables and independent independent of copies π and π' of permutations.

Proof. We apply Theorem 5.5 and get

$$\left(\mathbb{E}_{\varepsilon'\pi'} \mathbb{E}_{\varepsilon,\pi} \| \sum_{ij} \varepsilon_i \varepsilon_j' x_{\pi(i)\pi'(j)} \otimes y_{ij} \|_p \right) \sim_{c_p} n^{-\frac{1}{p}} \left(\mathbb{E}_{\varepsilon'\pi'} \sum_{il} \| \sum_j x_{l\pi'(j)} \otimes y_{ij} \|_p^p \right)^{\frac{1}{p}}$$

$$+ n^{-\frac{1}{2}} \left(\mathbb{E}_{\varepsilon'\pi'} \| \sum_{i,l,j} e_{1,il} \otimes x_{l\pi'(j)} \otimes y_{ij} \|_p^p \right)^{\frac{1}{p}} + n^{-\frac{1}{2}} \left(\mathbb{E}_{\varepsilon'\pi'} \| \sum_{i,l,j} e_{il,1} \otimes x_{l\pi'(j)} \otimes y_{ij} \|_p^p \right)^{\frac{1}{p}} .$$

We apply Theorem 5.5 for a second time to the first term and find

$$n^{-\frac{1}{p}} \left(\mathbb{E}_{\varepsilon'\pi'} \sum_{il} \| \sum_j x_{l\pi'(j)} \otimes y_{ij} \|_p^p \right)^{\frac{1}{p}} \sim_{c_p} n^{-\frac{2}{p}} \left(\sum_{l,k} \|x_{lk}\|_p^p \right)^{\frac{1}{p}} \left(\sum_{i,j} \|y_{ij}\|_p^p \right)^{\frac{1}{p}}$$

(6.1)

$$+ n^{-(\frac{1}{p}+\frac{1}{2})} \left(\sum_{il} \| \sum_{jk} e_{1,jk} \otimes x_{lk} \otimes y_{ij} \|_p^p \right)^{\frac{1}{p}} + n^{-(\frac{1}{p}+\frac{1}{2})} \left(\sum_{il} \| \sum_{jk} e_{jk,1} \otimes x_{lk} \otimes y_{ij} \|_p^p \right)^{\frac{1}{p}} .$$

Therefore, we have identified $c_1 = n^{-\frac{2}{p}} \left(\sum_{l,k} \|x_{lk}\|_p^p \right)^{\frac{1}{p}}$. We observe that $e_{1,jk}$ and $e_{1,k} \otimes e_{1,j}$ are conjugated by unitaries. This implies

$$\begin{aligned} \| \sum_{jk} e_{1,jk} \otimes x_{lk} \otimes y_{ij} \|_p &= \| (\sum_k e_{1,k} \otimes x_{lk}) \otimes (\sum_j e_{1,j} y_{ij}) \|_p \\ &= \| (\sum_k x_{lk} x_{lk}^*)^{\frac{1}{2}} \|_p \, \| (\sum_j y_{ij} y_{ij}^*)^{\frac{1}{2}} \|_p \,. \end{aligned}$$

Therefore, we find

$$c_3 \;=\; n^{-\frac{1}{p}-\frac{1}{2}} \left(\sum_l \|(\sum_k x_{lk} x_{lk}^*)^{\frac{1}{2}}\|_p^p \right)^{\frac{1}{p}} .$$

The same calculation yields $c_2 = n^{-\frac{1}{p}-\frac{1}{2}} \left(\sum_l \|(\sum_k x_{lk}^* x_{lk})^{\frac{1}{2}}\|_p^p \right)^{\frac{1}{p}}$. Applying Theorem 5.5 to the second and third term in (6.1) we may similarly identify the other constants c_4 up to c_9. They are always given by the same expression with the y_{ij}'s replaced by the x_{ij}'s and an additional normalizing factor depending on n. ■

Remark 6.2. Let us say that $(x_{ij}) \subset L_p(N)$ is a bi-symmetric basis if there exist a constant $C > 0$ such that

$$\| \sum_{ij} \varepsilon_i \varepsilon'_j a_{\pi(i)\pi(j)} x_{ij} \|_p \;\le\; C \| \sum_{ij} a_{ij} x_{ij} \|_p$$

holds for all scalar matrices (a_{ij}) with finite entries, all $\varepsilon_i = \pm 1$, $\varepsilon'_j = \pm 1$ and all permutation π and π'. For scalar coefficients the terms corresponding to c_2 and c_3 coincide. Also they coincide for c_4, c_5, for c_6, c_7 and c_8, c_9. Therefore we find the building blocks

$$\ell_p(\mathbb{N}^2), \ell_p(\ell_2), \ell_p(\ell_2)^{\perp}, S_p, \ell_2(\mathbb{N}^2)$$

and all possible intersections. Here $^{\perp}$ means that we interchange (i,j) to (ji). This yields a lengthy but complete characterization of all Banach spaces with bi-symmetric basis embedding into $L_p(N)$ for $2 < p < \infty$. On the operator space level, we have 9 building blocks

$$\begin{array}{ccccccccc}
S_p & & R_p(\mathbb{N}^2) & & & & C_p(\mathbb{N}^2) & & S_p^{\perp} \\
\downarrow & \swarrow\!\!\!\!\searrow & & & \swarrow\!\!\!\!\searrow & & & \swarrow\!\!\!\!\searrow & \downarrow \\
\ell_p(R_p) & & \ell_p(C_p)^{\perp} & & & & \ell_p(R_p)^{\perp} & & \ell_p(C_p) \\
\searrow & & \searrow & & & & \swarrow & & \swarrow \\
& & & & \ell_p(\mathbb{N}^2) & & & &
\end{array} .$$

Again one has to consider all possible intersection. Note that some of them simplify. The flashes indicate complete contractions, e.g. $S_p \subset \ell_p(R_p) \cap \ell_p(C_p)^{\perp}$ and $R_p(\mathbb{N}^2) \subset \ell_p(R_p) \cap \ell_p(R_p)^{\perp}$. We leave the task to calculate the number of spaces obtained in that way and further details to the interested reader.

Let E be a Banach space with a symmetric, normalized basis (e_k). The unitary ideal S_E is defined to be the closure of finite rank matrices with respect to the norm

$$\|a\|_{S_E} \;=\; \|\sum_k s_k(a) e_k\|_E \;,$$

where $(s_k(a)) = (\lambda_k(|a|))$ is the sequence of singular numbers.

Corollary 6.3. *Let $2 \leq p < \infty$ and E a symmetric space. Then S_E is isomorphic to $L_p(\mathcal{N})$ for some von Neumann algebra N if and only if $E = \ell_2$, $E = \ell_p$.*

Proof. If S_E is isomorphic to a subspace $L_p(N)$, then this is also true for the subspace spanned by the diagonal. According to Corollary 0.3 we have $E = \ell_2$ or $E = \ell_p$. In both cases $S_{\ell_p} = S_p$ and $S_{\ell_2} = S_2$ are complemented subspaces of S_p. ∎

Remark 6.4. Let X be an rearrangement invariant space on $(0,\infty)$ and N be a semifinite von Neumann algebra with a normal faithful trace. Then one can define $L_X(N,\tau)$ (see e.g. [DDdP]). The same argument as above shows that if $L_X(N,\tau)$ is isomorphic to a subspace $Y \subset L_p(\mathcal{M})$, then $L_X(N,\tau)$ is isomorphic to $L_p(N,\tau)$, $L_2(N,\tau)$ or $L_p(N,\tau) \cap L_2(N,\tau)$. In [Jun3] we show that indeed all these three spaces embed into $L_p(\mathcal{M})$ for a suitable choice of M. Thus the above characterizations 'extends' to the continuous case.

We conclude this section with a characterization of unitary ideals which embed $L_p(N)$ in the category of operator spaces. We need the following observation.

Lemma 6.5. *Let $2 \leq p < \infty$. Let $y_1, ..., y_n \in L_p(M)$. Then*

$$\left(\mathbb{E}\| \sum_{i=1}^n u_{ij} y_j \|_p^p \right)^{\frac{1}{p}} \sim_{c\sqrt{p}} \max\{ n^{-\frac{1}{2}} (\|(\sum_j y_j^* y_j)^{\frac12}\|_p, \|(\sum_j y_j y_j^*)^{\frac12}\|_p)\} \;.$$

Here the expectation is taken over the normalized Haar measure on the complex unitary group.

Proof. According to [MP], we may replace $(u_{ij})$by $\frac{g_{ij}}{\sqrt{n}}$ where g_{ij} are complex gaussian. In combination with the noncommutative Khintchine inequality (see [LPP], [Pis2]), this yields

$$\begin{aligned} \mathbb{E}\left(\| \sum_{i=1}^n u_{ij} y_j \|_p^p \right)^{\frac{1}{p}} &\sim_c \; n^{-\frac12} \mathbb{E}\left(\| \sum_{i=1}^n g_{ij} y_j \otimes x_i \|_p^p \right)^{\frac{1}{p}} \\ &\sim_{c\sqrt{p}} n^{-\frac12} (\|(\sum_j y_j^* y_j)^{\frac12}\|_p + \|(\sum_j y_j y_j^*)^{\frac12}\|_p) \end{aligned}$$

∎

Theorem 6.6. *Let $2 \le p < \infty$ and $(x_{ij}) \subset L_p(N)$ and $C > 0$ such that*

$$\|\sum_{ijkl} u_{ik}x_{kl}v_{lj} \otimes y_{ij}\|_p \;\le\; C\|\sum_{ij} x_{ij} \otimes y_{ij}\|_p$$

holds for all $y_{ij} \in L_p(M_m)$, $m \in \mathbb{N}$ and all unitary matrices u and v. Then there are constant $c_6, ..., c_9$ such that

$$\|\sum_{ij} x_{ij} \otimes y_{ij}\|_{L_p(N\otimes M)} \sim_{f_p(C)} c_6\|\sum_{i,j=1}^{n} e_{ij,1} \otimes y_{ij}\|_{L_p(M_{n^2}\otimes\mathcal{M})} + c_7\|\sum_{i,j=1}^{n} e_{1,ij} \otimes y_{ij}\|_{L_p(M_{n^2}\otimes\mathcal{M})}$$
$$+\, c_8\|\sum_{i,j=1}^{n} e_{ji} \otimes y_{ij}\|_{L_p(M_n\otimes\mathcal{M})} + c_9\|\sum_{i,j=1}^{n} e_{ij} \otimes y_{ij}\|_{L_p(M_n\otimes\mathcal{M})}\,.$$

Proof. Note that permutation matrices and the diagonal metrices with entries $\varepsilon_1, ..., \varepsilon_n$ are unitaries. This implies

$$\left(\mathbb{E}\|\sum_{ij} \varepsilon_k\varepsilon_l x_{\pi(i),\pi'(j)} \otimes (\sum_{kl} u_{ik}y_{ij}v_{lj})\|^p\right)^{\frac{1}{p}} \sim_{C^2} \|\sum_{ij} x_{ij} \otimes y_{ij}\|_p\,.$$

We first fix an integer $n \in \mathbb{N}$ and integrate with respect to the Haar in $\varepsilon, \varepsilon'$, π and π'. According to Proposition 6.1 we obtain 9 terms. The terms corresponding to c_6 to c_9 are invariant under unitary transformation from the right an the left. Let us consider the term corresponding to c_2. Using the unitary invariance of the column space and Lemma (6.5), we get

$$\left(\mathbb{E}\sum_{l=1}^{n}\|\sum_{i,j,k} e_{k,1} \otimes (u_{ik}y_{ij}v_{lj})\|_p^p\right)^{\frac{1}{p}} = \left(\mathbb{E}\sum_{l=1}^{n}\|\sum_{i}(\sum_{k} u_{ik}e_{k,1}) \otimes (\sum_{j} y_{ij}v_{lj})\|_p^p\right)^{\frac{1}{p}}$$
$$= \left(\mathbb{E}\sum_{l=1}^{n}\|\sum_{i} e_{i,1} \otimes (\sum_{j} y_{ij}v_{lj})\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{l=1}^{n}\mathbb{E}\|\sum_{j}(\sum_{i} e_{i,1} \otimes y_{ij})v_{lj}\|_p^p\right)^{\frac{1}{p}}$$
$$\sim_{c\sqrt{p}} n^{-\frac{1}{2}}\left(\sum_{l=1}^{n}\|\sum_{i,j} e_{j1} \otimes e_{i1} \otimes y_{ij}\|_p^p\right)^{\frac{1}{p}} + n^{-\frac{1}{2}}\left(\sum_{l=1}^{n}\|\sum_{i,j} e_{1j} \otimes e_{i1} \otimes y_{ij}\|_p^p\right)^{\frac{1}{p}}$$
$$= n^{\frac{1}{p}-\frac{1}{2}}\|\sum_{ij} e_{ij,1} \otimes y_{ij}\|_p + n^{\frac{1}{p}-\frac{1}{2}}\|\sum_{ij} e_{ij} \otimes y_{ij}\|_p\,.$$

In the finite dimensional case, we find additional contributions c_4 to c_9. In the infinite dimensional case these terms converge to 0 for $n \to \infty$. Indeed, from the proof of Proposition 6.1 we see that the constant $c_1, ..., c_9$ are uniformly bounded provided the (x_{ij}) are uniformly bounded. The uniform bound follows by applying the assumption to scalar coefficients. Therefore, we may pass to a subsequence such that $c_1(n), ..., c_9(n)$ converge to constant $c_1, ..., c_9$ such that $c_1 = c_2 = \cdots = c_5 = 0$. ■

Remark 6.7. The result shows that the building block for operator space unitary ideals embedding in $L_p(N)$ for $2 \leq p < \infty$ are

$$S_p, S_p^{\perp}, C_p(\mathbb{N}^2), R_p(\mathbb{N}^2) .$$

Since there are no trivial inclusions this amounts to 16 spaces obtained by intersections.

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