A decomposition trick

Definition 0.1 *$A \sim B$ if there is a bijection $\phi : A \to B$.*

Lemma 0.2 *Let A, B be sets such that*

a) $\dot{\bigcup}_n A \sim A$.

b) There is $B \sim A_1 \subset A$.

c) There is $A \sim B_1 \subset B$

Then $A \sim B$.

Proof: First, we note that $X \sim X_1$ and $Y \sim Y_1$ implies of course that $X \dot{\cup} Y \sim X_1 \dot{\cup} Y_1$. Then, we observe that since $\mathbb{N} \sim \mathbb{N}_0$, we have

$$A \dot{\cup} A \sim (\dot{\bigcup_n} A) \dot{\cup} A \sim \dot{\bigcup_{n \geq 0}} A \sim \dot{\bigcup_n} A \sim A \, .$$

Let us define $A_2 = A \setminus A_1$ and $B_2 = B \setminus B_1$. Then, we have

$$A = A_1 \dot{\cup} A_2 \quad \text{and} \quad B = B_1 \dot{\cup} B_2 \, .$$

This implies

$$B \dot{\cup} A \;=\; B_1 \dot{\cup} B_2 \dot{\cup} A \sim A \dot{\cup} B_2 \dot{\cup} A \sim A \dot{\cup} A \dot{\cup} B_2 \sim A \dot{\cup} B_2 \sim B_1 \dot{\cup} B_2 \sim B \, .$$

Moreover, $B \sim A_1$ implies

$$\begin{aligned} B \dot{\cup} A \;&\sim\; B \dot{\cup} \dot{\bigcup_n} A \sim B \dot{\cup} (\dot{\bigcup_n} A_1) \dot{\cup} (\dot{\bigcup_n} A_2) \\ &\sim\; \dot{\bigcup_{n \geq 0}} A_1 \dot{\cup} \dot{\bigcup_n} A_2 \\ &\sim\; \dot{\bigcup_n} A_1 \dot{\cup} \dot{\bigcup_n} A_2 \\ &\sim\; \dot{\bigcup_n} A \sim A \end{aligned}$$

Thus

$$B \sim B\dot{\cup}A \sim A\,.$$

The assertion is proved. □

Remark 0.3 *Since the Cantor set C is a subset of $[0,1]$, we may write*

$$[0,1] \;=\; C\dot{\cup}[0,1]\setminus C\,.$$

Since, there is a surjective map $f : C \to [0,1]$, we can apply the axiom of choice to find a function $\phi : [0,1] \to C$ such that $f(\phi(x)) = x$. Then ϕ has to be injective and thus

$$C \;=\; \phi([0,1])\dot{\cup}C \setminus \phi([0,1])\,.$$

Lemma 0.4 $[0,1) \sim \dot{\bigcup}_n [0,1)$.

Proof: Obviously $\dot{\bigcup}_n [0,1) \sim [0,\infty)$ and clearly $f(x) = 1-(1-x)^{-1}$ is a bijection between $[0,1)$ and $[0,\infty)$. □

Corollary 0.5 $[0,1] \sim [0,1)$ *and* $[0,1] \sim \bigcup_n [0,1]$.

Proof: $[0,1) \subset [0,1]$ and $[0,1] \sim [0,\frac{1}{2}] \subset [0,1)$ and $[0,1) \sim \dot{\bigcup}_n [0,1)$. Thus the first assertion follows from Lemma 0.2. Moreover,

$$[0,1] \sim [0,1) \sim \dot{\bigcup_n}[0,1) \sim \dot{\bigcup_n}[0,1]\,.$$

The assertion is proved. □

Conclusion 1): Let $A \subset [0,1]$ such that there is a surjection $f : A \to [0,1]$, then

$$A \sim [0,1]\,.$$

Lemma 0.6 *Let A be an infinite set, then*

$$A \sim \bigcup_n A\,.$$

Proof: Let $<$ be a well-ordering on A and $s(x) = \min\{y|y > x, y \neq x\}$ be the successor function. Let $\sim$ be the equivalence relation defined by $x \sim y$ if there exists $k \in \mathbb{N}_0$ such that $y = s^k(x)$ or $x = s^k(y)$. Since A is infinite, we see that $[x]$ has $\mathbb{N}$ many elements. Consider $B = A/\sim$. Then, we can find a bijection between A and $A/\sim \times\mathbb{N}$. Indeed, define, $h(n, [x]) = s^n(\min[x])$. Since, $\mathbb{N} \sim \bigcup_n \mathbb{N} = \mathbb{N} \times \mathbb{N}$, we get

$$A \sim B \times \mathbb{N} \sim B \times (\mathbb{N}^2) \sim \dot{\bigcup_n} B \times \mathbb{N} \sim \dot{\bigcup_n} A \,.$$

The assertion is proved. $\square$

Conclusion 2: If A and B are infinite sets and there exists an injection $f : A \to B$ and an injection $g : B \to A$, then

$$A \sim B \,.$$

Same with surjection (see Remark 0.3).