Exercise 2.7. Let $\varepsilon > 0$ be given. Show that there exists $\delta > 0$ with the following property. If A is a C^*-algebra and if a is an element in A such that $\|a - a^*\| \leqslant \delta$ and $\|a^2 - a\| \leqslant \delta$, then there is a projection p in A with $\|a - p\| \leqslant \varepsilon$. In other words, an element a in A, which is almost a projection, is close to a projection in A. [Hint: We need only consider the case where $\varepsilon < 1/2$. Put $b = (a + a^*)/2$. Show that the spectrum of b is contained in $[-\varepsilon, \varepsilon] \cup [1 - \varepsilon, 1 + \varepsilon]$ if $\|b - b^2\| \leqslant \varepsilon - \varepsilon^2$, and put $p = f(b)$ for a suitably chosen continuous function f.]

Show that for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If A is a C^*-algebra, B is a sub-C^*-algebra of A, and p is a projection in A such that $\|p - b\| \leqslant \delta$ for some b in B, then there is a projection q in B with $\|p - q\| \leqslant \varepsilon$.

Exercise 2.8. Let $\varepsilon > 0$ be given. Show that there exists $\delta > 0$ with the following property. If A is a unital C^*-algebra and a is an element in A such that $\|aa^* - 1_A\| \leqslant \delta$ and $\|a^*a - 1_A\| \leqslant \delta$, then there is a unitary element u in A with $\|a - u\| \leqslant \varepsilon$. In other words, an element a in A, which is almost unitary, is close to a unitary element in A. [Hint: Look at Proposition 2.1.8 and Paragraph 2.1.9.]

Show that for each $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If A is a unital C^*-algebra, B is a sub-C^*-algebra of A containing the unit of A, and u is a unitary element in A such that $\|u - b\| \leqslant \delta$ for some element b in B, then there is a unitary element v in B with $\|u - v\| \leqslant \varepsilon$.

Exercise 2.9. Let $\mathrm{Tr}\colon M_n(\mathbb{C}) \to \mathbb{C}$ be the standard trace given by

$$\mathrm{Tr}\begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} = \sum_{j=1}^{n} \alpha_{jj}.$$

Let p, q be projections in $M_n(\mathbb{C})$. Show that the following are equivalent:

(i) $p \sim q$,

(ii) $\mathrm{Tr}(p) = \mathrm{Tr}(q)$,

(iii) $\dim(p(\mathbb{C}^n)) = \dim(q(\mathbb{C}^n))$.

Use this to show that

$$\mathcal{D}(\mathbb{C}) \cong \{0, 1, 2, \dots\} = \mathbb{Z}^+,$$

when $\mathbb{Z}^+$ is equipped with the usual addition.

Show finally that the implications

$$\text{“}p \sim q \Rightarrow p \sim_u q\text{”} \quad \text{and} \quad \text{“}p \sim q \Rightarrow p \sim_h q\text{”}$$

hold for all projections p, q in $M_n(\mathbb{C})$.

Exercise 2.10. Let p, q be projections in $B(H)$, where H is an infinite dimensional separable Hilbert space.

(i) Show that $p \sim q$ if and only if $\dim(p(H)) = \dim(q(H))$.

(ii) Show that $p \sim_u q$ if and only if

$$\dim(p(H)) = \dim(q(H)) \quad \text{and} \quad \dim(p(H)^\perp) = \dim(q(H)^\perp).$$

This result is in Example 3.3.3 used to show that

$$\mathcal{D}(B(H)) \cong \{0, 1, 2, \dots, \infty\} = \mathbb{Z}^+ \cup \{\infty\},$$

where addition on $\mathbb{Z}^+ \cup \{\infty\}$ is the usual addition on $\mathbb{Z}^+$ and where $\infty + n = n + \infty = \infty$ for all n in $\mathbb{Z}^+ \cup \{\infty\}$.

Exercise 2.11. Show that $\mathcal{D}(\mathbb{C}\oplus\mathbb{C})$ is isomorphic to the additive semigroup $\mathbb{Z}^+ \oplus \mathbb{Z}^+$.

Exercise 2.12. Consider the short exact sequence

$$0 \longrightarrow C_0(\mathbb{R}^2) \xrightarrow{\ \varphi\ } C(\mathbb{D}) \xrightarrow{\ \psi\ } C(\mathbb{T}) \longrightarrow 0,$$

where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$, where ψ is the restriction mapping, and where φ is obtained by identifying $\mathbb{D} \setminus \mathbb{T}$ with $\mathbb{R}^2$. (You may replace $\mathbb{R}^2$ with $\mathbb{D} \setminus \mathbb{T}$ if you wish.) Let v in $C(\mathbb{T})$ be given by $v(z) = z$ for all z in $\mathbb{T}$.

(i) Show that v is unitary.

(ii) Show that v does not lift to a unitary in $C(\mathbb{D})$, i.e., there is no unitary u in $C(\mathbb{D})$ such that $\psi(u) = v$. [Hint: Use Brouwer's fixed point theorem which says that each continuous function $f\colon \mathbb{D} \to \mathbb{D}$ has a fixed point.]

(iii) Conclude that v does not belong to $\mathcal{U}_0(C(\mathbb{T}))$, and that there exist unitaries v_1, v_2 in $C(\mathbb{T})$ such that $v_1 \not\sim_h v_2$. Show that there is no self-adjoint element h in $C(\mathbb{T})$ for which $v = \exp(ih)$.