Math 406-History of Calculus- Homework 4

Due date September 23.

1) Consider Archimedes approximation for π with a regular polygon with n sides. Show that

$$t_n = \tan(\pi/n) \; ,$$
$$s_n = 2\sin(\pi/n) \; ,$$
$$o_n + m_n = \frac{\tan(\pi/n) + \sin(\pi/n)}{2}(1 - \cos(\pi/n)) \; .$$

What means that nt_n approaches π. Use this to determine

$$\lim_n n \sin(\pi/n) \; .$$

Solution: We consider two triangles, both with angle $\theta = \frac{2\pi}{2n}$, i.e. half the angle of the regular polygon. Then first triangle has three sides with length 1, $\cos(\theta)$ and $\sin(\theta)$. Then $\sin(\theta)$ is half the length of the inscribed triangle P_n^θ. Therefore the $s_n = 2\sin(\pi/n)$ and the Archimedian principle implies

$$n \sin(\frac{\pi}{n}) \;\le\; \pi \; .$$

The other triangle Q_n^θ contains the sector D^θ of the unit circle. The corresponding sides are r_n, t_n and 1. Here r_n is the radius of the circle in which Q_n is inscribed. Since both triangles are similar, we get

$$\frac{t_n}{1} \;=\; \frac{\sin(\theta)}{\cos(\theta)} \;=\; \tan(\frac{\pi}{n}) \; .$$

We also find $\frac{r_n}{1} = \frac{1}{\cos(\theta)} = \frac{1}{\cos(\pi/n)}$. again by the Archimedian principle we have

$$n \sin(\frac{\pi}{n}) \;\le\; \pi \;\le\; nt_n \;=\; n\frac{\sin(\frac{\pi}{n})}{\cos(\pi/n)} \; .$$

We have shown in class that $\lim_n \frac{1}{\cos(\pi/n)} = 1$. Therefore the squeeze theorem shows that

$$\lim_n n \sin(\frac{\pi}{n}) \;=\; \pi \; .$$

We should compare this with

$$\lim_{x \to 1} \frac{\sin(x)}{x}$$

which can be easily derived from l'Hopital's rule. Let us consider the last part. The part $m_n + o_n$ corresponds to the part between the line s_n and t_n. This is a trapezoid length $2t_n$, s_n and height $(1-\cos\theta)$ and this gives

$$m_n + o_n \;=\; \frac{2t_n + s_n}{2}(1-\cos(\theta)) \;=\; (\tan(\pi/n) + \cos(\pi/n))(1-\cos(\pi/n))\,.$$

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2) Complete the proof of $a(D(r)) = \frac{r}{2}|\partial D(r)|$. The missing case is $a(D(r)) > \frac{1}{2}r|\partial D(r)|$.
Assume that $a(D(r)) > \frac{1}{2}r|\partial D(r)|$. Then we can find $P_n \subset D(r)$ such that

$$a(P_n) > \frac{1}{2}r|\partial D(r)| > \frac{1}{2}r|\partial P_n| \;\geq\; a(P_n)\,.$$

This contradiction completes the proof. ■

3) Let C be cone with basis r and slanted height s. We want to show that

$$|\partial C_{up}| \;\geq\; s|\partial D(r)|\,.$$

Here $|\partial C_{up}|$ is the surface area of the upper part so that

$$|\partial C| \;=\; a(D(r)) + |\partial C_{up}|\,.$$

Proceed in the following steps

i) Assume that $|\partial C_{up}| < s|\partial D(r)|$. Find a regular polygon $P_n(r) \subset D(r)$ such that

$$|\partial C_{up}| < s|\partial P_n(r)|\,.$$

Indeed, we know that $|\partial P_n(r)|$ exhausts $|\partial D(r)|$, because we have completed No 2. Thus for $\alpha = \frac{|\partial C_{up}|}{s} < |\partial D(r)|$ we can find $P_n(r) \subset D(r)$ such that

$$\frac{|\partial C_{up}|}{s} < |\partial P_n(r)|\,.$$

Multiplying by s yields i).

ii) Observe that

$$|\partial C| < a(D(r)) + s|\partial P_n(r)|$$

find a regular polygon $P_m(r) \subset D(r)$ such that

$$|\partial C| < a(P_m(r)) + s|\partial P_n(r)|$$

Indeed, we have

$$|\partial C| \;=\; a(D(r)) + |\partial C_{up}| < a(D(r)) + s|\partial P_n(r)|$$

by i). This means

$$|\partial C| - s|\partial P_n(r)| < a(D(r))$$

By exhaustion, we can find m such that

$$|\partial C| - s|\partial P_n(r)| < a(P_m(r)) \; .$$

Hence we have

$$|\partial C| < s|\partial P_n(r)| + a(P_m(r)) \; .$$

iii) Join all the edges and conclude. Note that in both cases we can choose $P_n(r)$ and $P_m(r)$ be regular polygons obtained by successively adding midpoints. This means $n = 2^k$ and $m = 2^l$. We may take $j = \max k, l$, or simply the polygon P_s with contains all the points in P_n and all the the points in P_m. By the Archimedian principle we deuce that

$$|\partial C| < s|\partial P_s(r)| + a(P_s(r)) \; = \; |\partial \hat{P}_s(r)| \; ,$$

where $\hat{P}_s(r)$ is the pyramid with base area $P_s(r)$ together with the top point of the cone. Then $\hat{P}_s(r) \subset C$. By the Archimedian principle, we get

$$|\partial P_s(r)| \; \leq \; |\partial C|$$

because both gadgets are convex. This we reach the contradiction

$$|\partial C| < |\partial C| \; .$$

This means we must have

$$|\partial C_{up}| \; \geq \; s|\partial D(r)| \; .$$

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