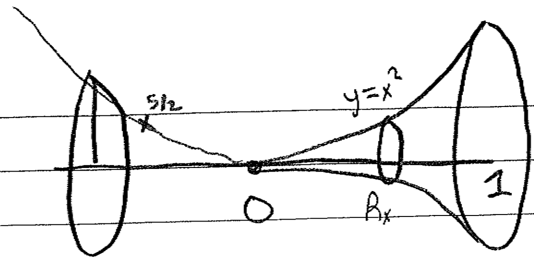
(2)



$z(x) = x^{5/2}$ $\quad$ $y(x) = x^2$

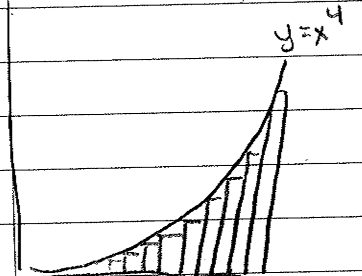
$a(R_x) = \pi y^2 = \pi x^4$

Lever Method

$$k\, a(Q_x) = x\, a(R_x) = \pi x^5 \Rightarrow k = 1$$

$$\Rightarrow k \text{ vol}(Q) = x_c \text{ vol}(R)$$

$$\Rightarrow x_c = k\frac{\text{vol}(Q)}{\text{vol}(R)} = \frac{\pi \int_0^1 x^5\, dx}{\pi \int_0^1 x^4\, dx}$$



$$\text{area} \geq 0\left(\tfrac{1}{n}\right) + \left(\tfrac{1}{n}\right)^4\left(\tfrac{1}{n}\right) + \left(\tfrac{2}{n}\right)^4\left(\tfrac{1}{n}\right) + \dots + \left(\tfrac{n-1}{n}\right)^4\left(\tfrac{1}{n}\right)$$

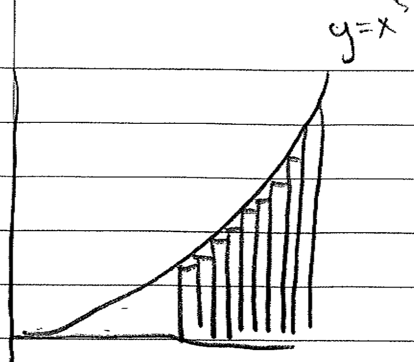
$$\leq \sum_{j=1}^{n} \left(\tfrac{j}{n}\right)^4\left(\tfrac{1}{n}\right)$$

$$\Rightarrow \left(\tfrac{1}{n^5}\right)\sum_{j=1}^{n} j^4 = \left(\tfrac{1}{n^5}\right)\left(\tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n\right)$$

$$= \frac{1}{5} + \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{30n^4}$$

$$= \frac{1}{5} \text{ as } n \text{ gets large}$$

$$\Rightarrow \int_0^1 x^4\, dx = \frac{1}{5}$$



$$\text{area} \geq \sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^5 \left(\frac{1}{n}\right)$$

$$\leq \sum_{i=1}^{n} \left(\frac{i}{n}\right)^5 \left(\frac{1}{n}\right)$$

$$\Rightarrow \left(\frac{1}{n^6}\right)\sum_{i=1}^{n} i^5 = \left(\frac{1}{n^6}\right)\left(\frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2\right)$$

$$= \frac{1}{6} + \frac{1}{2n} + \frac{5}{12n^2} - \frac{1}{12n^4}$$

$$= \frac{1}{6} \text{ as } n \text{ gets large}$$

$$\Rightarrow \int_0^1 x^5\,dx = \frac{1}{6}$$

$$\Rightarrow \quad x_c = \frac{\int_0^1 x^5\,dx}{\int_0^1 x^4\,dx} = \frac{1/6}{1/5} = \frac{5}{6}$$

$k=3$

$$(n+1)\sum_{i=1}^{n} i^{3} = \sum_{i=1}^{n} i^{4} + \sum_{p=1}^{n}\sum_{i=1}^{p} i^{3}$$

$$(n+1)\left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}\right) = \sum_{i=1}^{n} i^{4} + \sum_{p=1}^{n} \frac{p^4}{4}+\frac{p^3}{2}+\frac{p^2}{4}$$

$$(n+1)\left(n^4+2n^3+n^2\right) = 4\sum_{i=1}^{n} i^{4} + \sum_{p=1}^{n} p^4 + 2\sum_{p=1}^{n} p^3 + \sum_{p=1}^{n} p^2$$

$$(n+1)\left(n^4+2n^3+n^2\right) = 5\sum_{i=1}^{n} i^{4} + 2\left(\frac{n^4}{4}+\frac{2n^3}{4}+\frac{n^2}{4}\right) + \left(\frac{1}{6}\right)\left(2n^3+3n^2+n\right)$$

$$n^5+3n^4+3n^3+n^2 = 5\sum_{i=1}^{n} i^{4} + \left(\frac{1}{2}\right)n^4+n^3+\left(\frac{1}{2}\right)n^2+\left(\frac{1}{3}\right)n^3+\left(\frac{1}{2}\right)n^2+\left(\frac{1}{6}\right)n$$

$$n^5+\left(\frac{5}{2}\right)n^4+\left(\frac{5}{3}\right)n^3-\left(\frac{1}{6}\right)n = 5\sum_{i=1}^{n} i^{4}$$

$$\sum_{i=1}^{n} i^{4} = \left(\frac{1}{30}\right)\left(6n^5+15n^4+10n^3-n\right)$$

$k=4$

$$(n+1)\sum_{i=1}^{n} i^4 = \sum_{i=1}^{n} i^5 + \sum_{p=1}^{n}\sum_{i=1}^{p} i^4$$

$$(n+1)\left(\tfrac{1}{30}\right)\left(6n^5+15n^4+10n^3-n\right) = \sum_{i=1}^{n} i^5 + \sum_{p=1}^{n}\left(\tfrac{1}{30}\right)\left(6p^5+15p^4+10p^3-p\right)$$

$$(n+1)\left(6n^5+15n^4+10n^3-n\right) = \left(6n^6+21n^5+25n^4+10n^3-n^2-n\right)$$

$$\left(6n^6+21n^5+25n^4+10n^3-n^2-n\right) = 30\sum_{i=1}^{n} i^5 + \sum_{p=1}^{n} 6p^5+15p^4+10p^3-p$$

$$= 36\sum_{i=1}^{n} i^5 + 15\sum_{p=1}^{n} p^4 + 10\sum_{p=1}^{n} p^3 - \sum_{p=1}^{n} p$$

$$= 36\sum_{i=1}^{n} i^5 + \left(\tfrac{1}{2}\right)\left(6n^5+15n^4+10n^3-n\right) + \left(\tfrac{10}{4}\right)\left(n^4+2n^3+n^2\right) - \left(\tfrac{1}{2}\right)\left(n^2+n\right)$$

$$\left(12n^6+42n^5+50n^4+20n^3-2n^2-2n\right) = 72\sum_{i=1}^{n} i^5 + 6n^5+15n^4+10n^3-n+5n^4+10n^3+5n^2-n^2-n$$

$$\left(12n^6+42n^5+50n^4+20n^3-2n^2-2n\right) =$$

$$72\sum_{i=1}^{n} i^5 + \left(6n^5+20n^4+20n^3+4n^2-2n\right)$$

$$\left(12n^6+36n^5+30n^4-6n^2\right) = 72\sum_{i=1}^{n} i^5$$

$$\sum_{i=1}^{n} i^5 = \left(\tfrac{1}{12}\right)\left(2n^6+6n^5+5n^4-n^2\right)$$