K-Theory -Homework 3

Due Date: The week before Thanksgiving. All problems are on page 56ff

(1) 3.9
(2) 3.10
(3) 3.11

Exercise 3.7. Let A and B be C^*-algebras, and assume that we are given a *-homomorphism $\varphi_t\colon A \to B$ for each t in $[0,1]$. Show that the set of elements a in A for which the function $t \mapsto \varphi_t(a)$ is continuous is a sub-C^*-algebra of A.

Suppose that F is a generating subset of A, in other words, $A = C^*(F)$ (see Paragraph 1.1.2), and that the function $t \mapsto \varphi_t(x)$ is continuous for each x in F. Conclude that $t \mapsto \varphi_t(a)$ is continuous for each a in A.

Exercise 3.8. Show that an element p in $M_2(\mathbb{C})$ is a one-dimensional projection if and only if

$$p = \begin{pmatrix} t & \omega\sqrt{t(1-t)} \\ \overline{\omega}\sqrt{t(1-t)} & 1-t \end{pmatrix},$$

for some t in $[0,1]$ and some complex number ω of modulus 1. Use this to show that the space $G_{2,1}$ of all one-dimensional projections in $M_2(\mathbb{C})$ is homeomorphic to the 2-sphere S^2.

Exercise 3.9. Let $\mathbf{C}$ be a category, see Paragraph 3.2.1. Two objects A and B in $\mathcal{O}(\mathbf{C})$ are said to be isomorphic, written $A \cong B$, if there exist morphisms φ in $\mathrm{Mor}(A,B)$ and ψ in $\mathrm{Mor}(B,A)$ such that $\psi \circ \varphi = \mathrm{id}_A$ and $\varphi \circ \psi = \mathrm{id}_B$.

(i) Show that $\cong$ is an equivalence relation on $\mathcal{O}(\mathbf{C})$.

(ii) Suppose that N and N' are zero objects in $\mathbf{C}$, see Paragraph 3.2.1. Show that $N \cong N'$.

(iii) Suppose that $\mathbf{C}$ has a zero object N. Let $0_{N,A}$ and $0_{A,N}$ be the unique morphisms in $\mathrm{Mor}(A,N)$ and in $\mathrm{Mor}(N,A)$, respectively. For each pair of morphisms A and B in $\mathbf{C}$, set $0_{B,A} = 0_{B,N} \circ 0_{N,A}$ in $\mathrm{Mor}(A,B)$. Show that $0_{B,A}$ is independent of the choice of zero object N.

Exercise 3.10. Let R be a ring. For natural numbers m and n, the set of $m \times n$ matrices over R is denoted by $M_{m,n}(R)$ (or $M_n(R)$ in the case where $m = n$). An element e in R is *idempotent* if $e^2 = e$. We write $\mathcal{I}(R)$ for the set of idempotent elements in R, and we set

$$\mathcal{I}_n(R) = \mathcal{I}(M_n(R)), \qquad \mathcal{I}_\infty(R) = \bigcup_{n=1}^{\infty} \mathcal{I}_n(R).$$

Define the relation $\approx_0$ on $\mathcal{I}_\infty(R)$ as follows. Suppose that e belongs to $\mathcal{I}_n(R)$ and f to $\mathcal{I}_m(R)$. Then $e \approx_0 f$ if $e = ab$ and $f = ba$ for some elements a in $M_{n,m}(R)$ and b in $M_{m,n}(R)$.

(i) Suppose that $e \approx_0 f$, where e belongs to $\mathcal{I}_n(R)$ and f to $\mathcal{I}_m(R)$. Show that there are elements c in $M_{n,m}(R)$ and d in $M_{m,n}(R)$ such that $e = cd$, $f = dc$, $cdc = c$, and $dcd = d$. [Hint: Take $c = aba$ and $d = bab$, where a and b are as in the definition above.]

(ii) Show that $\approx_0$ is an equivalence relation on $\mathcal{I}_\infty(R)$.

(iii) Define an operation $\oplus$ on $\mathcal{I}_\infty(R)$ by

$$e \oplus f = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}, \qquad e, f \in \mathcal{I}_\infty(R).$$

Show that $e \oplus 0_n \approx_0 e$ for each e in $\mathcal{I}_\infty(R)$ and each natural number n, where 0_n is the zero element of $M_n(R)$.

(iv) Set $V(R) = \mathcal{I}_\infty(R)/\approx_0$. For each e in $\mathcal{I}_\infty(R)$, let $[e]_V$ denote the equivalence class of e in $V(R)$. Show that $\oplus$ induces an operation $+$ on $V(R)$ satisfying

$$[e]_V + [f]_V = [e \oplus f]_V, \qquad e, f \in \mathcal{I}_\infty(R).$$

(v) Show that $(V(R), +)$ is an Abelian semigroup.

In the case where R is a unital ring, $K_0(R)$ is defined to be the Grothendieck group of $(V(R), +)$.

In Exercise 3.11 it will be shown that this definition of K_0 agrees with the existing definition when A is a unital C^*-algebra.

Exercise 3.11. Let A be a C^*-algebra. The aim of this exercise is to show that the ring-theoretic definition of $K_0(A)$ given in Exercise 3.10 is essentially the same as the C^*-algebraic definition given in Definition 3.1.4. The terminology is the same as in Exercise 3.10.

(i) Show that, for every idempotent element e in A, there is a projection p in A with $e \approx_0 p$. [Hint: Observe that the element $h = 1+(e-e^*)(e^*-e)$ in $\widetilde{A}$ is invertible, and show that $eh = ee^*e = he$, $e^*h = e^*ee^* = he^*$. Set $p = ee^*h^{-1}$, and check that p is a projection in A satisfying $ep = p$ and $pe = e$.]

(ii) For projections p and q in A, show that $p \sim_0 q$ if and only if $p \approx_0 q$. [Hint: Take a and b in A with $p = ab$, $q = ba$, $a = aba$, and $b = bab$. Show that b^*b belongs to pAp and that $p \leqslant \|a\|^2 b^*b$. Deduce that $(b^*b)^{1/2}$

is invertible in pAp, take c in pAp such that $(b^*b)^{1/2}c = p = c(b^*b)^{1/2}$, and set $v = bc$. Show that $p = v^*v$ and $q = vv^*$; it may be helpful to verify that $qv = v$ first.]

(iii) Show that the map

$$[p]_{\mathcal{D}} \mapsto [p]_V, \quad \mathcal{D}(A) \to V(A),$$

is a well-defined semigroup isomorphism. [Hint: Observe that (i) and (ii) hold in any matrix algebra over A. To extend (ii) to projections in matrix algebras of different sizes, use Exercise 3.10 (iii).]

A somewhat related result can be found in Lemma 11.2.7.

Exercise 3.12. This exercise requires knowledge about von Neumann algebras. Let $\mathcal{M}$ be a von Neumann algebra factor of type II_1. Let τ be the unique normalized trace on $\mathcal{M}$. Show that

$$K_0(\tau)\colon K_0(\mathcal{M}) \to \mathbb{R}$$

is an isomorphism, so that $K_0(\mathcal{M}) \cong \mathbb{R}$. [Hint: Use the following two facts about II_1-factors. Two projections in a II_1-factor are equivalent if (and only if) they have the same trace, and $\{\tau(p) : p \in \mathcal{P}(\mathcal{M})\} = [0,1]$.]

Show that $K_0(\mathcal{M}) = 0$ if $\mathcal{M}$ is a factor of type II_∞ or of type III. (Factors of type I_∞ also have trivial K_0-group as shown in Example 3.3.3.)

Exercise 3.13. The purpose of this exercise is to verify the claim from Example 3.3.6 that the map $t \mapsto \varphi_t(f)$ is continuous for each f in $C(X)$, where X is a compact Hausdoff space, where $\alpha\colon [0,1] \times X \to X$ is a continuous map, and where $\varphi_t(f)(x) = f(\alpha(t,x))$.

Let t_0 in $[0,1]$ and $\varepsilon > 0$ be given. Set

$$W = \{(t,x) : |f(\alpha(t,x)) - f(\alpha(t_0,x))| < \varepsilon\}.$$

Show that there is $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \times X$ is contained in W, and conclude that $\|\varphi_t(f) - \varphi_{t_0}(f)\| \leqslant \varepsilon$ for all t in $(t_0 - \delta, t_0 + \delta)$.

Chapter 4

The Functor K_0

We extend the functor K_0 considered in Chapter 3 to a functor from the category of all C^*-algebras, unital or not. The K_0-group of a non-unital C^*-algebra is defined as a relative K-group, and it is shown that this definition is coherent with the definition of K_0 for unital C^*-algebras.

We give a standard picture of K_0 (a concrete description of the elements in $K_0(A)$). This new standard picture is less pedestrian than the standard picture for K_0 for unital C^*-algebras stated in Proposition 3.1.7, reflecting the greater complexity of K_0 in the non-unital case.

It is shown that K_0 is a functor which is half exact, split exact, and stable. Examples are given to show that K_0 is not exact.

4.1 Definition and functoriality of K_0

Definition 4.1.1 (The K_0-group of non-unital C^*-algebras). Let A be a non-unital C^*-algebra, and consider the associated split exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} \widetilde{A} \underset{\lambda}{\overset{\pi}{\rightleftarrows}} \mathbb{C} \longrightarrow 0 \tag{4.1}$$

obtained by adjoining a unit to A (see Paragraph 1.1.6). Define $K_0(A)$ to be the kernel of the homomorphism $K_0(\pi)\colon K_0(\widetilde{A}) \to K_0(\mathbb{C})$.

Observe that $K_0(A)$ is an Abelian group, being a subgroup of $K_0(\widetilde{A})$.

For each p in $\mathcal{P}_\infty(A)$ consider the equivalence class $[p]_0$ in $K_0(\widetilde{A})$. Since $K_0(\pi)([p]_0) = [\pi(p)]_0 = 0$, it follows that $[p]_0$ belongs to $K_0(A)$. In this way we obtain a map $[\,\cdot\,]_0\colon \mathcal{P}_\infty(A) \to K_0(A)$.