Math 441 Homework

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Problem 1. (p.74 # 32) Show that Proposition 23 need not be true if the integer variable n is replaced by a real variable t; that is, construct a family $\langle f_t \rangle$ of measurable real-valued functions on [0,1] such that for each x we have $\lim_{t\to 0} f_t(x) = 0$, but for some $\delta > 0$ we have $m^*\{x : f_t(x) > \frac{1}{2}\} > \delta$.

Solution 1. Let P_i be the sets in Section 4. First of all, since [0,1)=$\bigcup P_i$ and $\sum m^* P_i \geq m^*[0,1) = 1$, $m^* P_i \triangleq \delta > 0$. Fix x, then $\exists i$ such that $x \in P_i$. Then, for $t < 2^{-i-1}$, $f_t(x) \equiv 0$, which shows that $\lim_{t\to 0} f_t(x) = 0$ pointwise. Suppose, by way of contradiction, that there is a measurable set A such that f_t converges uniformly on $[0,1) \setminus A$, where $mA < \frac{\delta}{2}$. In other words, there exists T such that for $x \notin A$, $f_t(x) < \frac{1}{2}$ for all $t \leq T$. Consider P_i with $2^{-i} < T$. For any $x \in P_i$, we can find $t < T$ such that $x = 2^{i+1}t - 1$, that is, $f_t(x) = 1$ for $x \in P_i$. It means $P_i \subseteq A$, which implies $\delta = m^* P_i \leq m^* A = \frac{\delta}{2}$.

Problem 2. (p.93 # 12) Let g be an integrable function on a set E and suppose that $\langle f_n \rangle$ is a sequence of measurable functions such that $|f_n(x)| \leq g(x)$ a.e. on E. Then

$$\int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E \overline{\lim} f_n.$$

Solution 2. The second inequality is obvious. We need following claims.

Claim 1: If $\langle f_n \rangle$ is a sequence of nonnegative functions, then

$$\int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

Let $g_k = \inf_{i \geq k} f_i$, then $0 \leq g_1 \leq g_2 \leq \cdots$ and $g_k \to \underline{\lim} f_n$. By Fatou's lemma, $\int_E \underline{\lim} f_n \leq \underline{\lim}_k \int_E \inf_{i \geq k} f_i$. Since $\inf_{i \geq k} f_i \leq f_k$, from Proposition 8 $\int_E \inf_{i \geq k} f_i \leq \int_E f_k$. Thus

$$\int_E \underline{\lim}_n f_n \leq \underline{\lim}_k \int_E f_k = \underline{\lim}_n \int_E f_n. \qquad \square$$

Claim 2: Claim 1 holds for $\langle f_n \rangle$ a sequence of functions which are nonnegative almost everywhere!

For each n, there exists A_n such that $f_n \geq 0$ on $E \setminus A_n$ and $mA_n = 0$. Let $A = \bigcup A_n$, then $mA = 0$ and

$$\int_E \underline{\lim} f_n = \int_{E \setminus A} \underline{\lim} f_n \overset{\text{Claim1}}{\leq} \underline{\lim} \int_{E \setminus A} f_n = \underline{\lim} \int_E f_n. \qquad \square$$

Now, we apply Claim 2 to $\langle g + f_n \rangle$, then we have

$$\int_E \underline{\lim}(g + f_n) \leq \underline{\lim} \int_E (g + f_n). \qquad (*)$$

First of all, f_n is integrable($\because |f_n| \leq g$). Secondly, $\underline{\lim} f_n = \lim_{n \to \infty} \inf_{k \geq n} f_k$ is integrable from dominated convergence theorem($\because \inf_{k \geq n} f_k \leq |f_k| \leq g$). From $(*)$, we have

$$\int_E g + \int_E \underline{\lim} f_n = \int_E \underline{\lim}(g + f_n) \leq \int_E g + \underline{\lim} \int_E f_n,$$

that is,

$$\int_E \underline{\lim} f_n \leq \underline{\lim} \int_E f_n.$$

Similarly, applying Claim 2 to $\langle g - f_n \rangle$, we have

$$\overline{\lim} \int_E f_n \leq \int_E \overline{\lim} f_n.$$

Problem 3. (p.93 # 14)

a. Show that under the hypotheses of Theorem 17 we have $\int |f_n - f| \to 0$.

b. Let $\langle f_n \rangle$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

Solution 3.

a. Let $\tilde{f}_n = |f_n - f|$ and $\tilde{g}_n = g_n + |f|$, then $|\tilde{f}_n| = |f_n - f| \leq g_n + |f| = \tilde{g}_n$. $\tilde{g}_n \to g + |f|$ and $\int_E \tilde{g}_n = \int_E (g_n + |f|) \to \int_E g + \int_E |f| = \int_E (g + |f|)$, so applying Theorem 17, we obtain $\int_E |f_n - f| = \int_E |\tilde{f}_n| \to \int_E \lim |\tilde{f}_n| = 0$.

b.
($\Rightarrow$) $\left|\int_E |f_n| - \int_E |f|\right| = \left|\int_E (|f_n| - |f|)\right| \leq \int_E ||f_n| - |f|| \leq \int_E |f_n - f| \to 0$.
($\Leftarrow$) Apply part **a** above with $g_n = |f_n|$.

Problem 4. (p.94 # 18) Let f be a function of two variables $\langle x, t\rangle$ which is defined in the square $Q = \{\langle x, t\rangle : 0 \leq x \leq 1, 0 \leq t \leq 1\}$ and which is a measurable of x for each fixed value of t. Suppose that $\lim_{t\to 0} f(x,t) = f(x)$ and that for all t we have $|f(x,t)| \leq g(x)$, where g is an integrable function on [0,1]. Then

$$\lim_{t\to 0} \int f(x,t)dx = \int f(x)dx.$$

Show also that if the function $f(x,t)$ is continuous in t for each x, then

$$h(t) = \int f(x,t)dx$$

is a continuous function of t.

Solution 4. It is sufficient to show that for any sequence $\langle t_n\rangle$ with $t_n \to 0$,

$$\lim_{n\to\infty} \int f(x,t_n)dx = \int f(x)dx. \qquad (\star)$$

Let $f_n(x) = f(x,t_n)$, then $|f_n(x)| \leq g(x)$ and $\lim_{n\to\infty} f_n(x) = f(x)$. Thus ($\star$) follows immediately by Lebesgue's dominated convergence theorem. For the second part of the problem, we need to show that

$$\lim_{t\to t_0} \int f(x,t)dx = \int f(x,t_0)dx \qquad (\star\star)$$

for any $t_0 \in [0,1]$. Since $f(x,t)$ is continuous in t for each x, we have

$$\lim_{t\to t_0} f(x,t) = f(x,t_0).$$

Thus same argument as above works for this case.