Some solutions

Problem 1: Let

A) $\forall k,l\ :\ ks_1 < ls_2 \Leftrightarrow km_1 < lm_2$.

B) $s_1/s_2 = m_1/m_2$.

C) $\forall k,l\ :\ (ks_1 < ls_2 \text{and} km_1 < lm_2)\text{or}(ks_1 = ls_2 \text{and} km_1 = lm_2)\text{or}(ks_1 > ls_2 \text{and} km_1 > lm_2)$.

Then the statements $A)$, $B)$, $C)$ are equivalent.

PROOF. $A \Rightarrow C$: Assume A) and let $k, l \in \mathbb{N}$.

Case 1: $ks_1 < ls_2$: Then $A)$ implies $ks_1 < ls_2$ adn we are good.

Case 2: $ks_1 > ls_2$: similar.

Case 3: $ks_1 = ls_2$.

Case 3a): $km_1 < lm_2$: Apply case 1 with roles of m and s's interchanged. We get a contradiction.

Case 3b): $km_1 > lm_2$: Also leads to a contradiction.

Hence we must have $km_1 = lm_2$ as wanted in Eudoxos definition.

$C \Rightarrow A$: Assume C) and let $k, l \in \mathbb{N}$. Also assume $ks_1 < ls_2$. Then only the first statement in Eudoxos definition can be true and hence $km_1 < lm_2$.

Assume $km_1 < lm_2$: Repeat the argument with m and s's interchanged and we get $ks_1 < ls_2$.

$A \Rightarrow B$. Assume A). Assume $s_1/s_2 < m_1/m_2$. Between two positive rational numbers we can find a rational number. Hence there exists

$$s_1/s_2 < p/q < m_1/m_2 \ .$$

This means $s_1q < ps_2$ and $pm_2 < qm_1$. That contradicts A).

$B \Rightarrow A$. Assume B. Let $k, l \in \mathbb{N}$ such that $ks_1 < ls_2$. Then

$$km_1 \ = \ km_1/m_2m_2 \ = \ ks_1/s_2m_2 \ = \ \frac{ks_1}{ls_2}lm_2 < lm_2 \ .$$

The other case is similar. ∎

Problem 2: We need tow observations.

LEMMA 0.1. *Let $ls_1 < ks_2$ and $lm_1 \geq km_2$. Then there exists l', k' such that*

$$l's_1 < k's_2 \quad \text{and} \quad l'sm_1 > k'm_2 \ .$$

PROOF. Let N be a natural number such that

$$ls_1 < N(ks_2 - ls_1) \ .$$

Then we have

$$(N+1)ls_1 < Nks_2$$

and

$$(N+1)lm_2 > Nlm_2 \ \geq \ Nkm_2 \, .$$

Thus $l' = (N+1)l$ and $k' = Nk$ do the job. ■

LEMMA 0.2. *Let $ls_1 < ks_2$ and $lm_1 > km_2$. Then there exists $n \in \mathbb{N}$ such that*

$$ls_1 < kq_n \quad \textit{and} \quad lv_n > km_2 \, .$$

PROOF. We choose N such that

$$k(s_2 - q_1) < N(ks_2 - ls_1) \, .$$

Let $n > N$. Then

$$k(s_2 - q_n) \ \leq \ k2^{-n}(s_2 - q_1) < 2^{-n}N(ks_2 - ls_1) < (ks_2 - ls_1) \, .$$

Thus $ls_1 < kq_n$. We repeat the same argument and find for all $n' > N'$

$$l(m_1 - v_n) < 2^{-n'}N'(lm_1 - km_2) \, .$$

Taking the maximum of n and n' we see that

$$ls_1 < kq_n \quad \text{and} \quad lv_n > km_2$$

holds. ■

Let us now solve the problem. We want to show that $ks_1 < ls_2$ implies $km_1 < lm_2$. Let us assume to the contrary that $km_1 \ \geq \ lm_2$. We apply the first Lemma and replace k, l by k', l' such that

$$k's_1 < l's_2 \quad \text{and} \quad k'm_1 > l's_2 \, .$$

Then we apply the second Lemma and find n such that

$$k's_1 < l'q_n \quad \text{and} \quad k'v_n > l's_2 \, .$$

This implies

$$k'p_n < l'q_n \quad \text{and} \quad k'v_n > l'w_n \, .$$

By assumption $(p_n : q_n) = (v_n : w_n)$. Thus allows us to deduce from $k'p_n < l'q_n$ that also

$$k'v_n < l'w_n \, .$$

Thus we reach the contradiction

$$k'v_n < l'w_n < k'v_n \, .$$

■