Check up -problems

(1) Show that a bounded monotone sequence of real numbers is converging.

(2) Calculate the eigenvalues of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(3) Formulate an important result from your undergraduate analysis course.

(4) Formulate an important result from your undergraduate linear algebra course.

Practice problems for the final

(1) In the following problem you use $D = C[0,1]$ and $X = L_1[0,1]$ the completion of D with respect to the metric

$$d_1(f,g) \;=\; \int_0^1 |f-g|dt \;.$$

Show that the function $u : D \to \mathbb{R}$ given by

$$u(f) \;=\; \int_0^1 f(t)dt$$

has a unique extension on X. Let $g \in C[0,1]$ such that

$$\sup_s |g(s)| \le \alpha < 1$$

Is there a fixpoint f in D for $T(f)(s) = g(s)f(s)$? What happens if we use

$$g(s) \;=\; \begin{cases} \frac{1}{2} & 0 \le s \le \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} < s \le 1 \end{cases} \;.$$

(Then $T(D) \subset L_1$.)

Solution: We only have to show that u is uniformly continuous. For this it suffice to show that u is Lipschitz. Indeed,

$$|u(f)-u(h)| \;=\; |\int_0^1 [f(t)-h(t)]dt| \;\le\; \int_0^1 |f(t)-h(t)|dt \;=\; d_1(f,h) \;.$$

Thus the unique extension principle yields the claim. Not we consider $T : D \to D$ defined by $T(f) = fg$. Note that

$$d_1(T(f),T(h)) \;\le\; \int |g(t)||f(t)-h(t)|dt \;\le\; \sup|g(t)|d_1(f,h) \;\le\; \alpha d_1(f,h) \;.$$

Thus T is Lipschitz with constant $\le \alpha$. In particular, T admits an extension $T : L_1 \to L_1$ which is also Lipschitz with constant $\le \alpha$. By he contraction principle there exists a unique fixpoint f for T. This means

$$f(t)g(t) \;=\; f(t)$$

or equivalently $f(t)(1-g(t)) \;=\; 0$. Thus $f(t) \;=\; 0$ is the only fixpoint-how boring. The argument in the second case is similar, we find $f \in L_1$ such that $T(f) = f$ but the fixpoint is unique and $f(t) = 0$ is a solution. So this is the only solution.

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculate e^{xA} and solve the system of equations

$$\vec{y}'(x) = A(\vec{y}(x)) \quad , \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

Determine the JNF without finding a basis.

Solution: If we consider $(A-1)$ then it is east to see that $\dim\ker(A-1) = 1$, $\dim\ker(A-1)^2 = 2$, $\dim\ker(A-1)^3 = 3$ is $\dim\ker(A-1)^4 = 4$. Thus we have one Jordan block of size 4. We consider

$$B = (A-1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $B^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and and $B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This yields

$$e^{xA} = e^x \left(\sum_{k=0}^{\infty} \frac{x^k B^k}{k!} \right) = e^t(id + xB + \frac{x^2}{2}B^2 + \frac{x^3}{3!}B^3 .$$

This means

$$e^{xA} = e^x \begin{pmatrix} 1 & x & x+\frac{x^2}{2} & x+x^2+\frac{x^3}{6} \\ 0 & 1 & x & x+\frac{x^2}{2} \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

For the system, we have $\vec{y}(t) = e^{xA}y(0)$-Plug it in.

(3) In this problem we want to solve the differential equation

$$\frac{\partial^2}{\partial x \partial t} F(x,t) = F(x,t) \quad , \quad F(0,t) = f(t) \tag{0.1}$$

for some continuous function f on $[0,1]$. Recall that for any Banach space X and a continuous linear map $T : X \to X$. The solution to

$$y'(x) \;=\; T(y(x)) \quad y(0) = y_0$$

is given by $y(x) \;=\; e^{xT}y_0$. (We have used that for matrices). The trick here is to introduce

$$T(f)(t) \;=\; \int_0^t f(s)ds\,.$$

(1) Show that $g = T(f)$ if and only if $g(0) = 0$ and $g'(t) = f(t)$.
(2) Show that $T : C[0,1] \to C[0,1]$ is continuous.
(3) Let $f \in C[0,1]$. Let $n \in \mathbb{N}$. Differentiate the function

$$g_n(t) \;=\; \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds\,.$$

(4) Show that $T^n(f) = g$ if and only if $g^{(n)} = f$ and

$$g(0) = g'(0) = \cdots = g^{(n-1)}(0) = 0\,.$$

Use this to show that

$$T^n(f) \;=\; \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds\,.$$

(5) Show that

$$F(x,t) \;=\; f(t) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds$$

is a solution of (**??**). (Hint F is a such a solution iff $\frac{d}{dx}F(x,t) \;=\; \int_0^t F(x,s)ds$ and we have just found a solution to that, isn't it.)

Solution: a) is just the fundamental theorem of calculus. b) We have

$$\|T(f)\| \;=\; \sup_t |\int_0^t f(s)ds| \;\le\; \sup_t t \sup_s |f(s)| \;\le\; \sup_s |f(s)|\,.$$

This means $\|T\| \le 1$. (T is Lipschitz with constant 1.) c) is an application of the chain rule (we assume $n > 1$)

$$G(t,r) \;=\; \int_0^t \frac{(r-s)^{n-1}}{(n-1)!} f(s)ds$$

$\frac{\partial G}{\partial t} \;=\; 0$ and $\frac{\partial G}{\partial r} = \int_0^t \frac{(r-s)^{n-2}}{(n-2)!} f(s)ds$. Now, we use

$$g_n(t) \;=\; G(t,t)$$

and deduce the assertion. d) That is induction and we note from c) that g_n satisfies the requirements, thus $T^n(f) = g_n$. e) We known that

$$y(x) \;=\; e^{xT}(y_0)$$

is the unique solution to

$$y'(x) \;=\; T(y(x)) \quad , \quad y(0) = y_0 \; .$$

We put $y_0 = f \in C[0,1]$ and deduce that

$$y(x) \;=\; e^{xT}(f)$$

is a solution. We define $F(x,t) \;=\; e^{xT}(f)(t)$. Since point evaluation is continuous we deduce

$$\frac{\partial}{\partial x} F(x,t) \;=\; T(F(x,t)) \;=\; \int_0^t F(x,s) ds \; .$$

We differentiate another time with respect to t and get

$$\frac{\partial^2}{\partial t \partial x} F(x,t) \;=\; F(x,t) \; .$$

There you go.

(4) Consider $Y = \ell_2$ and $F : \ell_2 \times \ell_2 \to \ell_2$ given by

$$F((x_n),(y_n)) \;=\; y_n - x_n^2 y_n - \frac{2}{n^4} \; .$$

We consider the couple $(x^0, y^0) = ((\frac{2}{n^2}), (0))$.

(1) Calculate $D_1 F$ and $D_2 F$. It is easy to see that these maps are continuous (try).

(2) Show that every linear map $L : \ell_2 \to \ell_2$ given by $L(h_n) \;=\; (a_n h_n)$ such that $C = \sup_n |a_n|^{-1}$ is finite satisfies

$$\|L^{-1}\| \;\leq\; C \; .$$

(3) Show that $D_2 F(x^0, y^0)$ is invertible.

(4) Show that there exists a $\delta > 0$ and function $f : B(x^0, \delta) \to \ell_2$ such that

$$F(x, f(x)) = 0$$

for all $x \in B(x^0, \delta)$. Calculate

$$f'(x^0) \in L(\ell_2) \; .$$

Solution: We consider

$$
\begin{aligned}
F((x_n+h_n),(y_n+k_n)) - F(x_n,y_n)_n \;&=\; k_n-(x_n+h_n)^2(y_n+k_n)+x_n^2y_n \\
= k_n - [(x_n^2+h_n^2+2x_nh_n)(y_n+k_n)] + x_n^2y_n \\
= k_n - [(x_n^2y_n+h_n^2y_n+2x_ny_nh_n)+(x_n^2k_n+h_n^2k_n+2x_nh_nk_n)] + x_n^2y_n \\
= k_n - 2x_ny_nh_n - x_n^2k_n - (h_n^2y_n+h_n^2k_n+2x_nh_nk_n)\ .
\end{aligned}
$$

Note that

$$
\|(h_n^2y_n)\|_2 \;\le\; \sup_n |y_n|(\sum_n |h_n|^4)^{\frac{1}{2}} \;\le\; \sup_n |y_n|(\sum_n |h_n|^2)^2\ .
$$

Similarly,

$$
\begin{aligned}
\|(2x_nh_nk_n)\|_2 &\le \sup_n |x_n|(\sum_n |h_nk_n|^2)^{\frac{1}{2}} \;\le\; \sup_n |x_n|[(\sum_n |h_n|^4)^{1/2}(\sum_n |k_n|^4)^{1/2}]^{1/2} \\
&\le \sup_n |x_n|\|(h_n)\|_2\|(k_n)\|_2 \\
&\le \sup_n |x_n|\frac{\|(h_n)\|_2^2+\|(k_n)\|_2^2}{2}\ .
\end{aligned}
$$

This is fine, too. The last term is similar. Thus F is differentiable and

$$
D_1F((x_n)(y_n))(h_n) \;=\; (-2x_ny_nh_n)_n
$$

and

$$
D_2F((x_n)(y_n))(k_n) \;=\; ((1-x_n^2)k_n)_n\ .
$$

(2) is obvious the inverse map is given by $L(h_n)=(a_n^{-1}h_n)$ and

$$
\|(a_n^{-1}h_n)\|_2 \le \sup_n |a_n^{-1}|\|(h_n)\|_2\ .
$$

For (3) we note that $a_n = 1-\frac{2}{n^2}$ is bounded away from 0. Finally (4) is easy

$$
f'(x^0)(h_n) \;=\; (a_n^{-1}(-\frac{2}{n^2}0h_n)) \;=\; 0\ .
$$

Thus $f(x^0)=0$.

Final 595-Transition

Name:

(1) Let $X \neq \emptyset$ be a metric space, $0 < \alpha < 1$ and $T : X \to X$ such that

$$d(T(x), T(y)) \ \le \ \alpha d(x, y) \ .$$

Show that there exists a sequence (x_n) such that

$$\lim_n d(T(x_n), x_n) \ = \ 0 \ .$$

(Hint: If you like abstract space you can argue with completion. That is not necessary though).

(2) Let

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the Jordan Normal form and e^{tA}. Solve the system of equations

$$\begin{aligned} y_1' &= y_1 + 3y_3 \\ y_2' &= y_2 + 2y_3 \\ y_3' &= y_3 \end{aligned}$$

with $y_1'(0) = y_2'(0) = y_3'(0) = 1$. (I know this can be done by hand-but I don't want you to make to difficult computations).

(3) Let $h \in C[0,1]$ be continuous function.

(a) Show that $T(f)(t) = \int_0^t h(s)f(s)ds$ is a continuous linear map such that $g = T(f)$ if and only if

$$g'(t) = h(t)f(t) .$$

(b) Show that there exists a function $F : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that

$$\frac{\partial^2}{\partial x \partial t} F(x,t) = h(t)F(x,t) .$$

(You may freely interchange integration and differentiation in this problem).

(4) Consider $Y = \ell_2$ (if you prefer finite dimensions work with $Y = \mathbb{R}^m$), and

$$F((x_n),(y_n))_n \;=\; y_n - (x_n + y_n)^2$$

Show that there exists a $\delta > 0$ and differentiable function $f : B(0,\delta) \to \ell_2$ such that

$$F(x, f(x)) \;=\; 0$$

Show that $f'(0) = 0$. (Hint: You may use that $\|D_{\lambda_n} : \ell_2 \to \ell_2\| = \sup_n |\lambda_n|$ holds for every diagonal operator. You may also use that if D_1F and D_2F are continuous, then F is continuously differentiable.)

Final 595-Transition

(1) Let $X \neq \emptyset$ be a metric space, $0 < \alpha < 1$ and $T : X \to X$ such that

$$d(T(x), T(y)) \;\leq\; \alpha d(x, y)\;.$$

Show that there exists a sequence (x_n) such that

$$\lim_n d(T(x_n), x_n) \;=\; 0\;.$$

(Hint: If you like abstract space you can argue with completion. That is not necessary though).

Proof: We consider an arbitrary point x_0 and $x_n = T^n(x_0)$. Note that

$$d(x_{n+1}, x_n) \;\leq\; \alpha d(x_n, x_{n-1}) \;\leq\; \alpha^n d(x_1, x_0)\;.$$

Since $\alpha < 1$ the assertion follows.

(2) Let

$$A \;=\; \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Determine the Jordan Normal form and e^{tA}. Solve the system of equations

$$\begin{aligned} y_1' &= y_1 + 3y_3 \\ y_2' &= y_2 + 2y_3 \\ y_3' &= y_3 \end{aligned}$$

with $y_1'(0) = y_2'(0) = y_3'(0) = 1$. (I know this can be done by hand-but I don't want you to make to difficult computations).

Solution: We consider $B = A - 1$ and find

$$B^2 \;=\; 0$$

Thus, we find the JNF as

$$A \cong \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\;.$$

For e^{tA} we get

$$e^{tA} = e^t e^{tB} = e^t \begin{pmatrix} 1 & 0 & 3t \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{pmatrix} .$$

Thus the solution is given by

$$\vec{y}(t) = e^t \begin{pmatrix} 1+3t \\ 1+2t \\ 1 \end{pmatrix} .$$

(3) Let $h \in C[0,1]$ be continuous function.

(a) Show that $T(f)(t) = \int_0^t h(s)f(s)ds$ is a continuous linear map such that $g = T(f)$ if and only if $g(0) = 0$ and

$$g'(t) = h(t)f(t) .$$

(b) Show that there exists a function $F : \mathbb{R} \times [0,1] \to \mathbb{R}$ such that

$$\frac{\partial^2}{\partial x \partial t} F(x,t) = h(t)F(x,t) .$$

(I am not asking for an explicit solution)

Solution: If

$$g(t) = \int_0^t h(s)f(s)ds$$

and f and h are continuous, then

$$g'(t) = h(t)f(t) .$$

Conversely, if h and f are continuous and $g' = hf$, then we deduce from the fundamental theorem

$$g(t) - g(0) = \int_0^t g'(s)ds = \int_0^t h(s)f(s)ds$$

Thus $g(0) = 0$ implies the assertion. For the second part, we show that $T : C[0,1] \to C[0,1]$ defined above is continuous. Indeed,

$$\begin{aligned} \|T\| &= \sup_{\|f\|\le 1} \sup_{0\le t\le 1} |\int_0^t h(s)f(s)ds| \\ &\le \sup_{\|f\|\le 1} \sup_{0\le t\le 1} t \sup_{0\le s\le 1} |h(s)| \sup_{0\le s\le 1} |f(s)| \le \|g\| . \end{aligned}$$

Thus for any initial condition $f \in C[0,1]$ we see that

$$y(x) \;=\; e^{xT}(f)$$

solves the ODE

$$y'(x) \;=\; T(y(x))\,.$$

Since pointwise evaluation is continuous, we deduce that $F(x,t) = y(x)(t)$ satisfies

$$\frac{\partial}{\partial x}F(x,t) \;=\; h(t)\int_0^t F(x,s)ds\,.$$

This yields $\frac{\partial}{\partial x}F(x,0) = 0$ and another differentiation yields

$$\frac{\partial^2}{\partial t\partial x}F(x,t) \;=\; h(t)F(x,t)\,.$$

There you go. (The numerical solution may use the exponential series).

(4) Consider $Y = \mathbb{R}^m$, and

$$F((x_n),(y_n))_n \;=\; y_n - (x_n+y_n)^2$$

Show that there exists a $\delta > 0$ and differentiable function $f : B(0,\delta) \to Y$ such that

$$F(x,f(x)) \;=\; 0$$

Show that $f'(0) \;=\; 0$.

Solution: We consider the function

$$f(x,y) \;=\; y - (x+y)^2$$

Then $\frac{\partial f}{\partial x}f(x,y) \;=\; -2(x+y)$ and $\frac{\partial f}{\partial y}f(x,y) \;=\; 1-2(x+y)$. Thus we try to

$$D_1F((x_n),(y_n))((h_n)) \;=\; -2(x_n+y_n)h_n$$

and

$$D_2F((x_n),(y_n))((h_n)) \;=\; (1-2(x_n+y_n))h_n\,.$$

In order to check that $F' = (D_1F, D_2)$ is the derivative, we consider

$$\|F((x_n+h_n),(y_n+k_n)) - F((x_n),(y_n)) - (D_1(F((x_n),(y_n))(h_n) - (D_2(F((x_n),(y_n))(k_n)\|$$

$$= \left(\sum_n |y_n+k_n-(x_n+h_n+y_n+k_n)^2-(y_n-(x_n+y_n)^2)\right.$$

$$\left. +2(x_n+y_n)h_n - k_n + 2(x_n+y_n)k_n|^2\right)^{\frac{1}{2}}$$

$$= \left(\sum_n | - (x_n^2 + h_n^2 + y_n^2 + k_n^2 + 2x_n(h_n + k_n) + 2y_n(h_n + k_n) + 2x_n y_n) \right.$$

$$\left. +(x_n^2 + y_n^2 + 2x_n y_n) + 2(x_n + y_n)h_n - k_n + 2(x_n + y_n)k_n|^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_n | - k_n^2 - h_n^2|^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_n |k_n|^2 \right)^{\frac{1}{2}} \sup_n |k_n| + \left(\sum_n |h_n|^2 \right)^{\frac{1}{2}} \sup_n |h_n|$$

$$\leq (\|(k_n)\| + \|(h_n)\|)^2 .$$

Thus we get differentiability. For $((x_n), (y_n)) = 0$ we get $D_2F = Id$. Hence it is invertible. The inverse function theorem provides the answer with

$$f'((x_n))(h_n) = -(\frac{-2(x_n + y_n)}{1 - 2(x_n + y_n)} h_n) .$$

Hence $f'(x) = 0$. ■