

Practice problems for the final

(1) In the following problem you use $D = C[0, 1]$ and $X = L_1[0, 1]$ the completion of D with respect to the metric

$$d_1(f, g) = \int_0^1 |f - g| dt .$$

Show that the function $u : D \rightarrow \mathbb{R}$ given by

$$u(f) = \int_0^1 f(t) dt$$

has a unique extension on X . Let $g \in C[0, 1]$ such that

$$\sup_s |g(s)| \leq \alpha < 1$$

Is there a fixpoint f in D for $T(f)(s) = g(s)f(s)$? What happens if we use

$$g(s) = \begin{cases} \frac{1}{2} & 0 \leq s \leq \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} < s \leq 1 \end{cases} .$$

(Then $T(D) \subset L_1$.)

Solution: We only have to show that u is uniformly continuous. For this it suffice to show that u is Lipschitz. Indeed,

$$|u(f) - u(h)| = \left| \int_0^1 [f(t) - h(t)] dt \right| \leq \int_0^1 |f(t) - h(t)| dt = d_1(f, h) .$$

Thus the unique extension principle yields the claim. Now we consider $T : D \rightarrow D$ defined by $T(f) = fg$. Note that

$$d_1(T(f), T(h)) \leq \int |g(t)| |f(t) - h(t)| dt \leq \sup |g(t)| d_1(f, h) \leq \alpha d_1(f, h) .$$

Thus T is Lipschitz with constant $\leq \alpha$. In particular, T admits an extension $T : L_1 \rightarrow L_1$ which is also Lipschitz with constant $\leq \alpha$. By the contraction principle there exists a unique fixpoint f for T . This means

$$f(t)g(t) = f(t)$$

or equivalently $f(t)(1 - g(t)) = 0$. Thus $f(t) = 0$ is the only fixpoint-how boring. The argument in the second case is similar, we find $f \in L_1$ such that $T(f) = f$ but the fixpoint is unique and $f(t) = 0$ is a solution. So this is the only solution.

(2) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculate e^{xA} and solve the system of equations

$$\vec{y}'(x) = A(\vec{y}(x)) \quad , \quad \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} .$$

Determine the JNF without finding a basis.

Solution: If we consider $(A - 1)$ then it is east to see that $\dim \ker(A - 1) = 1$, $\dim \ker(A - 1)^2 = 2$, $\dim \ker(A - 1)^3 = 3$ is $\dim \ker(A - 1)^4 = 4$. Thus we have one Jordan block of size 4. We consider

$$B = (A - 1) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Then } B^2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \text{ This yields}$$

$$e^{xA} = e^x \left(\sum_{k=0}^{\infty} \frac{x^k B^k}{k!} \right) = e^t (id + xB + \frac{x^2}{2} B^2 + \frac{x^3}{3!} B^3) .$$

This means

$$e^{xA} = e^x \begin{pmatrix} 1 & x & x + \frac{x^2}{2} & x + x^2 + \frac{x^3}{6} \\ 0 & 1 & x & x + \frac{x^2}{2} \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

For the system, we have $\vec{y}(t) = e^{xA}y(0)$ -Plug it in.

(3) In this problem we want to solve the differential equation

$$\boxed{\text{pde}} \quad (0.1) \quad \frac{\partial^2}{\partial x \partial t} F(x, t) = F(x, t) \quad , \quad F(0, t) = f(t)$$

for some continuous function f on $[0, 1]$. Recall that for any Banach space X and a continuous linear map $T : X \rightarrow X$. The solution to

$$y'(x) = T(y(x)) \quad y(0) = y_0$$

is given by $y(x) = e^{xT}y_0$. (We have used that for matrices). The trick here is to introduce

$$T(f)(t) = \int_0^t f(s)ds .$$

- (1) Show that $g = T(f)$ if and only if $g(0) = 0$ and $g'(t) = f(t)$.
- (2) Show that $T : C[0, 1] \rightarrow C[0, 1]$ is continuous.
- (3) Let $f \in C[0, 1]$. Let $n \in \mathbb{N}$. Differentiate the function

$$g_n(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds .$$

- (4) Show that $T^n(f) = g$ if and only if $g^{(n)} = f$ and

$$g(0) = g'(0) = \dots = g^{(n-1)}(0) = 0 .$$

Use this to show that

$$T^n(f) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds .$$

- (5) Show that

$$F(x, t) = f(t) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds$$

is a solution of (0.1). (Hint F is a such a solution iff $\frac{d}{dx}F(x, t) = \int_0^t F(x, s)ds$ and we have just found a solution to that, isn't it.)

Solution: a) is just the fundamental theorem of calculus. b) We have

$$\|T(f)\| = \sup_t \left| \int_0^t f(s)ds \right| \leq \sup_t t \sup_s |f(s)| \leq \sup_s |f(s)| .$$

This means $\|T\| \leq 1$. (T is Lipschitz with constant 1.) c) is an application of the chain rule (we assume $n > 1$)

$$G(t, r) = \int_0^t \frac{(r-s)^{n-1}}{(n-1)!} f(s)ds$$

$\frac{\partial G}{\partial t} = 0$ and $\frac{\partial G}{\partial r} = \int_0^t \frac{(r-s)^{n-2}}{(n-2)!} f(s)ds$. Now, we use

$$g_n(t) = G(t, t)$$

and deduce the assertion. d) That is induction and we note from c) that g_n satisfies the requirements, thus $T^n(f) = g_n$. e) We know that

$$y(x) = e^{xT}(y_0)$$

is the unique solution to

$$y'(x) = T(y(x)) \quad , \quad y(0) = y_0 .$$

We put $y_0 = f \in C[0, 1]$ and deduce that

$$y(x) = e^{xT}(f)$$

is a solution. We define $F(x, t) = e^{xT}(f)(t)$. Since point evaluation is continuous we deduce

$$\frac{\partial}{\partial x} F(x, t) = T(F(x, t)) = \int_0^t F(x, s) ds .$$

We differentiate another time with respect to t and get

$$\frac{\partial^2}{\partial t \partial x} F(x, t) = F(x, t) ds .$$

There you go.

(4) Consider $Y = \ell_2$ and $F : \ell_2 \times \ell_2 \rightarrow \ell_2$ given by

$$F((x_n), (y_n)) = y_n - x_n^2 y_n - \frac{2}{n^4} .$$

We consider the couple $(x^0, y^0) = ((\frac{2}{n^2}), (0))$.

- (1) Calculate $D_1 F$ and $D_2 F$. It is easy to see that these maps are continuous (try).
- (2) Show that every linear map $L : \ell_2 \rightarrow \ell_2$ given by $L(h_n) = (a_n h_n)$ such that $C = \sup_n |a_n|^{-1}$ is finite satisfies

$$\|L^{-1}\| \leq C .$$

- (3) Show that $D_2 F(x^0, y^0)$ is invertible.
- (4) Show that there exists a $\delta > 0$ and function $f : B(x^0, \delta) \rightarrow \ell_2$ such that

$$F(x, f(x)) = 0$$

for all $x \in B(x^0, \delta)$. Calculate

$$f'(x^0) \in L(\ell_2) .$$

Solution: We consider

$$\begin{aligned}
F((x_n + h_n), (y_n + k_n)) - F(x_n, y_n)_n &= k_n - (x_n + h_n)^2(y_n + k_n) + x_n^2 y_n \\
&= k_n - [(x_n^2 + h_n^2 + 2x_n h_n)(y_n + k_n)] + x_n^2 y_n \\
&= k_n - [(x_n^2 y_n + h_n^2 y_n + 2x_n y_n h_n) + (x_n^2 k_n + h_n^2 k_n + 2x_n h_n k_n)] + x_n^2 y_n \\
&= k_n - 2x_n y_n h_n - x_n^2 k_n - (h_n^2 y_n + h_n^2 k_n + 2x_n h_n k_n).
\end{aligned}$$

Note that

$$\|(h_n^2 y_n)\|_2 \leq \sup_n |y_n| \left(\sum_n |h_n|^4 \right)^{\frac{1}{2}} \leq \sup_n |y_n| \left(\sum_n |h_n|^2 \right)^2.$$

Similarly,

$$\begin{aligned}
\|(2x_n h_n k_n)\|_2 &\leq \sup_n |x_n| \left(\sum_n |h_n k_n|^2 \right)^{\frac{1}{2}} \leq \sup_n |x_n| \left[\left(\sum_n |h_n|^4 \right)^{1/2} \left(\sum_n |k_n|^4 \right)^{1/2} \right]^{1/2} \\
&\leq \sup_n |x_n| \| (h_n) \|_2 \| (k_n) \|_2 \\
&\leq \sup_n |x_n| \frac{\| (h_n) \|_2^2 + \| (k_n) \|_2^2}{2}.
\end{aligned}$$

This is fine, too. The last term is similar. Thus F is differentiable and

$$D_1 F((x_n)(y_n))(h_n) = (-2x_n y_n h_n)_n$$

and

$$D_2 F((x_n)(y_n))(k_n) = ((1 - x_n^2)k_n)_n.$$

(2) is obvious the inverse map is given by $L(h_n) = (a_n^{-1} h_n)$ and

$$\|(a_n^{-1} h_n)\|_2 \leq \sup_n |a_n^{-1}| \| (h_n) \|_2.$$

For (3) we note that $a_n = 1 - \frac{2}{n^2}$ is bounded away from 0. Finally (4) is easy

$$f'(x^0)(h_n) = (a_n^{-1} (-\frac{2}{n^2} 0 h_n)) = 0.$$

Thus $f(x^0) = 0$.