

Transition course -hw2

Due date: Monday, September 13

- (1) Let $1 \leq p < \infty$. Show that $\ell_p = \{(x_n) : \left(\sum_n |x_n|^p\right)^{\frac{1}{p}} \text{ is a complete metric spaces with respect to}$

$$d_p(x, y) = \left(\sum_n |x_n - y_n|^p\right)^{\frac{1}{p}} .$$

(Hint: Use $d_p(x, y) = \lim_m \left(\sum_{n \leq m} |x_n - y_n|^p\right)^{\frac{1}{p}}$ in the proof of the triangle inequality.)

- (2) Use the result from the lecture for a short proof of the fact

$$\ell_\infty = \{(x_n) : \sup_n |x_n| < \infty\}$$

is complete with respect to $d(x, y) = \sup_n |x_n - y_n|$.

- (3) Show that $x_n = \sum_{j \leq n} p^j$ is a Cauchy sequence in (\mathbb{Z}, dd_p) . Show that there is no limit $x \in \mathbb{Z}$. (Hint use the unique decomposition $x = \sum a_k p^k$, $a_k \in \{0, \dots, p-1\}$ for positive integers.)
- (4) A function $f : X \rightarrow Y$ is uniformly continuous of

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X (d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon) .$$

Show that $f : (0, 1) \rightarrow \mathbb{R}$, $f(x) = 1/x$ is not uniformly continuous.

- (5) The space $C_b(X, \mathbb{R})$ is also a vector space over \mathbb{R} . Find n -linearly independent elements for
- (a) $X = \{1, \dots, n\}$ with the discrete metric.
 - (b) $X = [0, 1]$ with the usual metric. (Hint: polynomials?)