

Transition-hw3

**Due date:** September 27.

- (1) Show the discrete Chebychev inequality: Let  $(a_n)$  be positive numbers such that

$$S = \sum_n a_n < \infty$$

is finite. Let  $\lambda > 0$ . Show that the set  $A_\lambda = \{n \in \mathbb{N} : a_n \geq \lambda\}$  has not more than  $S/\lambda$  many elements.

- (2) Let  $A_1, \dots, A_k$  be subset of  $\{1, \dots, n\}$  such that

$$\sum_k |A_k| < n$$

Show there is an element in  $\bigcup_{j=1, \dots, k} A_j^c$ . ( $A_k^c = \{1, \dots, n\} \setminus A_k$  is the complement.)

- (3) Show that the countable union of sets of measure 0 has measure 0.  
 (4) We want to show that a bounded function is Riemann integrable if and only if the set of discontinuity points has measure 0. Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a function such that

$$\sup_{x \in [0, 1]} |f(x)| \leq B.$$

We define

$$\omega_\delta(x) = \sup\{|f(y) - f(x)| : |x - y| < \delta, |x - z| < \delta\}.$$

- (a) Show that  $\omega_\delta(x) \leq 2B$ .  
 (b) Let  $I = [a, b]$  be an interval and  $a + \delta \leq x \leq b - \delta$ . Show that

$$\omega_\delta(x)(b - a) \leq (\sup_I f - \inf_I f)(b - a).$$

- (c) Let  $N \in \mathbb{N}$  and consider the two partitions

$$\pi = \langle 0, 1/N, 2/N, \dots, N - 1/N, 1 \rangle$$

and

$$\pi' = \langle 0, 1/2N, 3/2N, \dots, 1 - 1/2N, 1 \rangle.$$

We define  $I_k = [\frac{k-1}{N}, \frac{k}{N}]$  ( $k = 1, \dots, N$ ) and  $I'_k = [\frac{k-1}{N} + \frac{1}{2N}, \frac{k}{N} + \frac{1}{2N}]$  ( $k = 1, \dots, N - 1$ ). The Lemma above suggest to consider for  $I = [a, b]$  the interval

$$I(\delta) = [a + \delta, b - \delta]$$

(for  $a = 0$  we will use  $I(\delta) = [0, b - \delta]$  and for  $b = 1$   $I(\delta) = [a + \delta, 1]$  why?). Show that

$$[0, 1] \subset \bigcup_{k=1}^N I_k(\delta) \cup \bigcup_{k=1}^{N-1} I'_k(\delta) .$$

(If you make a picture this is obvious).

- (d) Let  $A_{\delta,k} = \{x \in [0, 1] : \omega_\delta(x) \geq \frac{1}{k}\}$ . Let  $N \in \mathbb{N}$  such that  $\delta < 1/4N$ . Show that the set  $A_{\delta,k}$  is contained in intervals  $J_1, \dots, J_m$  such that

$$\sum_j |J_j| \leq k (\bar{S}(f, \pi) - \underline{S}(f, \pi)) + k (\bar{S}(f, \pi') - \underline{S}(f, \pi')) .$$

- (e) Use the Cauchy-criterion for the Darboux integral to show that for a Darboux- integrable function

$$\bigcap_{\delta} A_{\delta,k}$$

has measure 0.

- (f) Show that for a Darboux- integrable function the set of non-continuity points has measure 0.