Transition-hw6

Due date: October 25

(1) Let $T : V \to V$ be a linear map and $W \subset V$ such that $T(W) \subset W$. We denote by $q : V \to V/W$ the quotient map.

(a) Show that there is a linear map $\hat{T} : V/W \to V/W$ such that $qT = \hat{T}q$.

(b) Now assume that V is finite dimensional. We assume that W has a basis $S = \{s_1, ..., s_k\}$ such that

$$T(s_j) \;=\; \sum_{i \leq j} \lambda_{i,j} s_i \; .$$

(This says that the matrix of the map $T_W : W \to W$, $T_W(x) = T(x)$ has a matrix which is upper diagonal). We also assume that $\hat{T} : V/W \to V/W$ has a basis B such that

$$\hat{T}(b_k) \;=\; \sum_{l \leq k} a_{l,k} b_l \; .$$

Show that V has a basis such that the matrix of T is upper diagonal.

(c) Show that every matrix over an algebraically closed field is similar to an upper diagonal matrix. (Hint: Such a matrix has an eigenvector).

(2) (a) We consider $V = C(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$. Let $r > 0$ and $T(f)(t) \;=\; f(t + r)$. Find all the eigenvalues of T.

(b) We consider $V = C_0(\mathbb{R})$ the space of continuous functions f on $\mathbb{R}$ such that $\lim_{t\to\infty} f(t) = 0$. Let $r > 0$ and $T(f)(t) \;=\; f(t + r)$. Find all the eigenvalues of T.

(c) We consider $V = C[0, 1]$ and $n \in \mathbb{N}$. We define the linear map $T(f)(t) \;=\; f(t\dot{+}(\frac{1}{n}))$ where

$$s\dot{+}t \;=\; \begin{cases} s + t & \text{if } s + t \leq 1 \\ s + t - 1 & \text{if } s + t > 1 \; . \end{cases}$$

Find the eigenvalues of T.

(d) We consider $V = C[0, 1]$ and θ an irrational number. Use the fact that the sequence $\theta_n = \underbrace{\theta\dot{+} \cdots \dot{+}\theta}_{n \text{ times}}$ is dense in $[0, 1]$ in order to determine the eigenvalues of T.