Transition-hw7

Due date: November 1

(1) Calculate the Jordan normal form and appropriate bases of the following matrices.

i) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

ii) $B = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$

(2) Let us recall that $\det : (F^n)^n \to F$ is a map which satisfies the following conditions

i) $\det(v_1 + \lambda w_1, v_2, ..., v_n) = \det(v_1, ..., v_n) + \lambda \det(w_1, ..., v_n)$

ii) $\det(v_1, ..., v_{j-1}, v_j, v_{j+1}., v_n) = (-1)\det(v_j, ..., v_{j-1}, v_1, v_{j+1}., v_n)$,

iii) $\det(e_1, ..., e_n) = 1$

for all $w_1 \in F^n$, $\lambda \in F$, $v_1, ...v_n \in F^n$. Here $e_1, ..., e_n$ are the standard unit vectors. For a matrix $A = [a_{ij}]$ we define the column vectors $v_j = (a_{ij})_{i=1,..,n}$ and

$$\det(A) = \det(v_1, ..., v_n) .$$

In the following you may use that there is only one map $\det : (F^n)^n = \mathcal{M}_n \to F$ satisfying the conditions $i) \to iii)$.

(a) Show that $\det(AB) = \det(A)\det(B)$.

(b) For a permuation $\pi : \{1, ..., n\}$, we define a linear map $T_\pi(e_i) = e_{\pi(i)}$. Let A_π be the cooresponding matrix. We denote the group of permutation by S_n. Show that $\varepsilon : S_n \to F$ defined by

$$\varepsilon(\pi) = \det(A_\pi)$$

is a group homomorphism.

(c) Let $i \neq j$. Show that for a cycle (ij) (which interchanges i and j) we have

$$\varepsilon((ij)) \; = \; -1 \, .$$

(d) Show thet every permutation π may be written as a product of cycles (hint: use induction) and that for $\pi = (i_1 j_1) \cdots (i_m j_m)$ we have

$$\varepsilon(\pi) \; = \; (-1)^m \, .$$

(One can actually show that every permutation is a product of neighbouring cycles.)